Summary of Lecture 1

Degrees of freedom	Equation of motion
Gravitational potential $\psi(\eta, \vec{x})$	Einstein 00: $\phi, \psi \leftrightarrow \delta$
Scale factor distortion $\phi(\eta, \vec{x})$	Einstein ij: $(\phi - \psi) \longrightarrow \sigma$
CDM density/velocity $\delta_\mathrm{c}(\eta, \vec{x})\,, \theta_\mathrm{c}(\eta, \vec{x})$	continuity δ_c' + Euler θ_c'
Baryon density/velocity $\delta_\mathrm{b}(\eta, \vec{x})\,, \theta_\mathrm{b}(\eta, \vec{x})$	Continuity δ_b' + Euler θ_b' (incl. Thomson)
Photons $f_\gamma(\eta, \vec{x}, p, \hat{n})$	Boltzmann $\frac{d}{d\eta} f_\gamma =$ Thomson
[Neutrinos $f_\nu(\eta, \vec{x}, p, \hat{n})$]	[Boltzmann $\frac{d}{d\eta} f_\nu = 0$]

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Photons $\cancel{f_\gamma(\eta,\vec{x},p,\hat{n})}$ $\Theta(\eta,\vec{x},\hat{n})$	Boltzmann $\cancel{\frac{d}{d\eta}f_\gamma} =$ Thomson

$$\Theta' + \hat{n}\cdot\vec{\nabla}\Theta - \phi' + \hat{n}\cdot\vec{\nabla}\psi = -\Gamma_\gamma\left(\hat{n}\cdot(\vec{v}_\gamma - \vec{v}_{\rm b}) + \text{higher multipoles}\right)$$

$$\delta_\gamma' + \frac{4}{3}\theta_\gamma - 4\phi' = 0$$

$$\theta_\gamma' + k^2\left(-\frac{1}{4}\delta_\gamma + \sigma_\gamma\right) - k^2\psi = \tau'(\theta_\gamma - \theta_\mathrm{b})$$

$$\Theta_l' - \frac{kl}{2l+1}\Theta_{l-1} + \frac{k(l+1)}{2l+1}\Theta_{l+1} = \tau'\Theta_l \quad \forall l \geq 2$$

Degrees of

Gravitational pote

Scale factor disto

CDM density/velocity $\delta_\mathrm{c}(\eta,\vec{x})$, $\theta_\mathrm{c}(\eta,\vec{x})$ | continuity δ_c' + Euler θ_c'

Baryon density/velocit $\theta_\mathrm{b}(\eta,\vec{x})$ | Continuity δ_b' + Euler θ_b' (incl. Thomson)

$\Theta_\ell(\eta,\vec{k})$

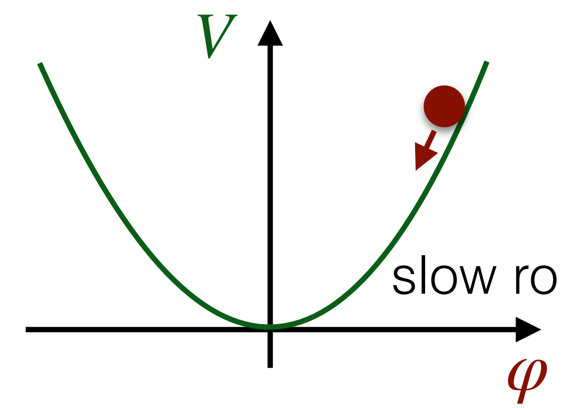
Photons $\cancel{f_\gamma(\eta,\vec{x},p,\hat{n})}$ | Boltzmann $\cancel{\frac{d}{d\eta}f_\gamma}$ = Thomson

$\Theta(\eta,\vec{x},\hat{n})$

$$\Theta' + \hat{n}\cdot\vec{\nabla}\Theta - \phi' + \hat{n}\cdot\vec{\nabla}\psi = -\Gamma_\gamma\Big(\hat{n}\cdot(\vec{v}_\gamma - \vec{v}_\mathrm{b}) + \text{higher multipoles}\Big)$$

Stochastic theory of cosmological perturbations

Initial conditions



Canonical single-field inflation guarantees:

A. stochastic perturbations with independent Fourier modes

B. gaussian statistics for each Fourier mode / each d.o.f.

$\Rightarrow$ described by variance(wavenumber) = power spectrum

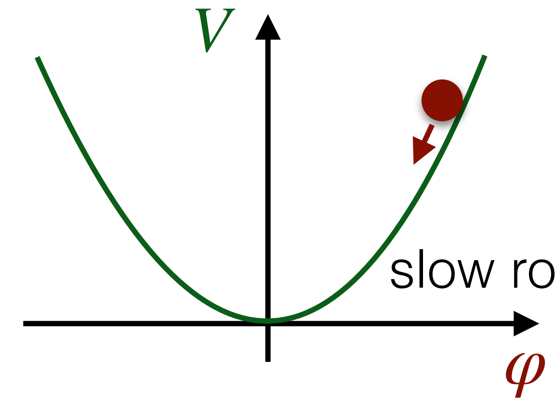
C. for each Fourier mode, all d.o.f. related to each other (fully correlated) on super-Hubble scales: “adiabatic initial conditions”

e.g. during RD:

$$-2\psi = -2\phi \underset{\substack{\uparrow \\ \text{Einstein eq.}}}{=} \boxed{\delta_\gamma = \delta_\nu = \frac{4}{3}\delta_{\rm b} = \frac{4}{3}\delta_{\rm c}} \underset{\substack{\uparrow \\ \text{Einstein eq.}}}{=} \text{constant}$$

(Comes from $A(\eta,\vec{x}) = \bar{A}(\eta + \delta\eta(\vec{x})) = \bar{A}(\eta) + \underbrace{\bar{A}'(\eta)\,\delta\eta(\vec{x})}_{\substack{\text{perturbation } \delta A(\eta,\vec{x}) \\ \text{in adiabatic case}}}$)

Primordial power spectrum



Canonical single-field inflation guarantees:

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C. for each Fourier mode, all d.o.f. related to each other (fully correlated) on super-Hubble scales: "adiabatic initial conditions"

⇒ need power spectrum for single degree of freedom, e.g. curvature perturbation $\mathcal{R} \equiv \phi - \frac{a'}{a}\frac{v_{\rm tot.}}{a^2}$ in Newt. Gauge

⇒ Primordial spectrum: $\langle \mathscr{R}(\eta_i, \vec{k}) \mathscr{R}^*(\eta_i, \vec{k}') \rangle = \delta_D(\vec{k}' - \vec{k}) \; P_{\mathscr{R}}(k)$

D. Power law, nearly scale-invariant spectrum: $P_{\mathscr{R}}(k) = \frac{2\pi^2}{k^3} A_s \left(\frac{k}{k_*} \right)^{n_s - 1}$

Transfer functions

For each Fourier mode $\vec{k}$:

- all perturbations → system of linear coupled differential equations
- adiabatic ICs → single constant of integration $\mathcal{R}(\eta_{\rm ini},\vec{k})$
- $\forall A \in \{\phi, \psi, \delta_X, \theta_X, \Theta_\ell, \ldots\}$

$$A(\eta, \vec{k}) = T_A(\eta, k)\, \mathscr{R}(\eta_i, \vec{k})$$

stochastic Fourier mode

stochastic IC

Deterministic solution of e.o.m. normalised to $\mathscr{R}$ =1

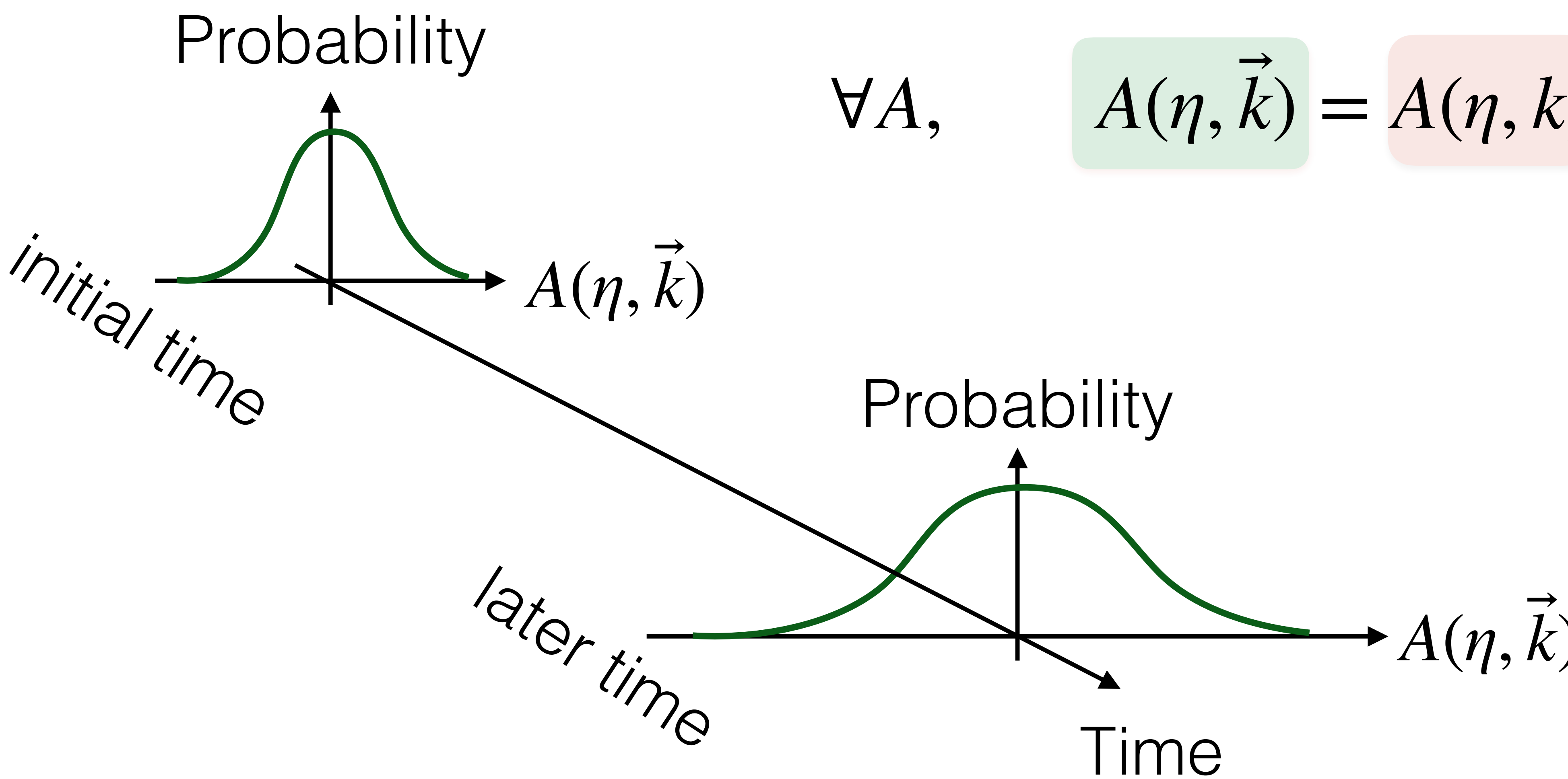
= transfer function of A

Isotropic background ⇒ depends only on k

⇒ denoted later as $A(t, k)$

Linear transport of probability

$$\forall A, \qquad A(\eta, \vec{k}) = A(\eta, k)\, \mathscr{R}(\eta_i, \vec{k})$$



Linearity of solutions ⇒ probability shape always preserved
(standard model: Gaussian)
⇒ variance evolves like square of transfer function

Power spectrum

Adiabatic initial conditions

$\Rightarrow$ for <u>any</u> perturbation at <u>any</u> time:

$$\langle A(\eta,\vec{k})\, A^*(\eta,\vec{k}')\rangle = A(\eta,k)\, A^*(\eta,k')\, \langle \mathscr{R}(\eta_{\rm i},\vec{k})\, \mathscr{R}^*(\eta_{\rm i},\vec{k}')\rangle$$

$$= \boxed{|A(\eta,k)|^2 P_{\mathscr{R}}(k)}\ \delta_D(\vec{k}-\vec{k}')$$

transfer function of A

power spectrum $P_A(\eta,k)$ of A at η

primordial curvature spectrum

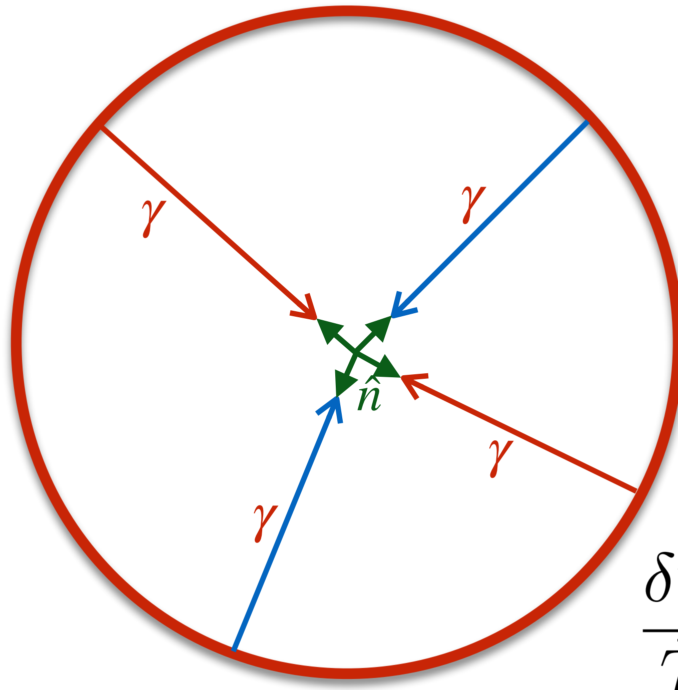
Spectrum of temperature anisotropies

Temperature multipoles

$g(\eta)$ very peaked at $\eta_{\rm dec}$

$\Downarrow$

last scattering sphere



$$\frac{\delta T}{\bar{T}}(\hat{n}) = \Theta(\eta_0, \vec{o}, -\hat{n}) = \sum_{lm} a_{lm} Y_{lm}(\hat{n})$$

inversion + Fourier + Legendre $\Rightarrow$ $a_{lm} = (-i)^l \int \frac{d^3\vec{k}}{2\pi^2} Y_{lm}(\hat{k}) \Theta_l(\eta_0, \vec{k})$

stochastic, Gaussian ⟵ stochastic, Gaussian

correlation/variance $\Rightarrow$ $\langle a_{lm} a^*_{l'm'} \rangle = \delta^K_{ll'} \delta^K_{mm'} \left[\frac{2}{\pi} \int dk\, k^2 \Theta_l^2(\eta_0, k) \; P_{\mathscr{R}}(k) \right]$

photon transfer function

primordial spectrum

Temperature power spectrum

Defined as: $$C_l = \langle a_{lm} a^*_{lm} \rangle = \frac{2}{\pi} \int dk\, k^2 \Theta_l^2(\eta_0, k)\, P_{\mathscr{R}}(k)$$

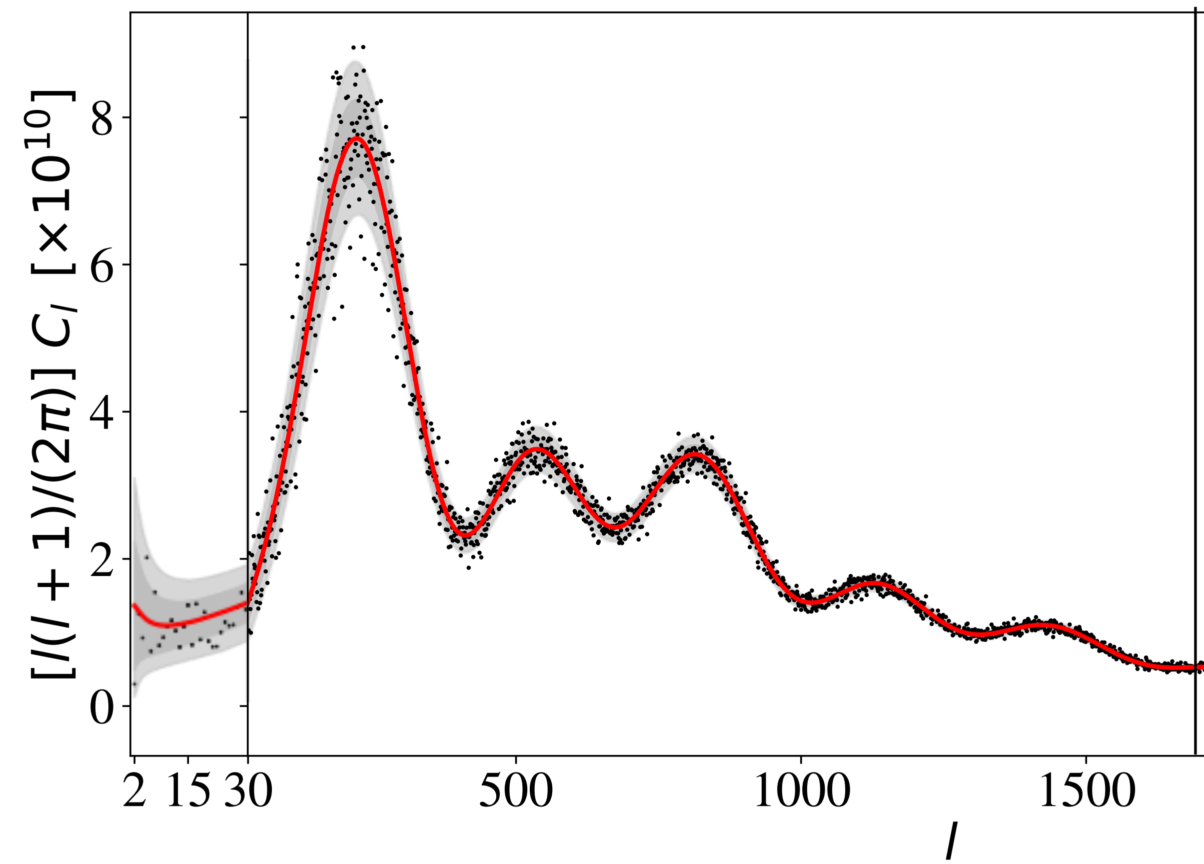
photon transfer function

primordial spectrum

theory $\longleftrightarrow$ observations

Estimator: $$\hat{C}_l(a_{lm}) \equiv \frac{1}{2l+1} \sum_{-l \leq m \leq l} |a_{lm}|^2$$

Cosmic variance: $$\langle (\hat{C}_l - C_l)^2 \rangle = \frac{2}{2l+1} C_l^2$$



Physics of temperature anisotropies

“Line-of-sight” integral in Fourier space

Boltzmann hierarchy ⇒ formal solution Zaldarriaga & Harari astro-ph/9504085:

$$\Theta_l(\eta_0, \overset{(\rightarrow)}{k}) = \int_{\eta_{\rm ini}}^{\eta_0} d\eta \, \{ g \, (\Theta_0 + \psi) \; j_l(k(\eta_0 - \eta))$$

$$+ \, g \, k^{-1} \theta_{\rm b} \; j_l'(k(\eta_0 - \eta))$$

$$+ \, e^{-\tau} (\phi' + \psi') \; j_l(k(\eta_0 - \eta)) \}$$

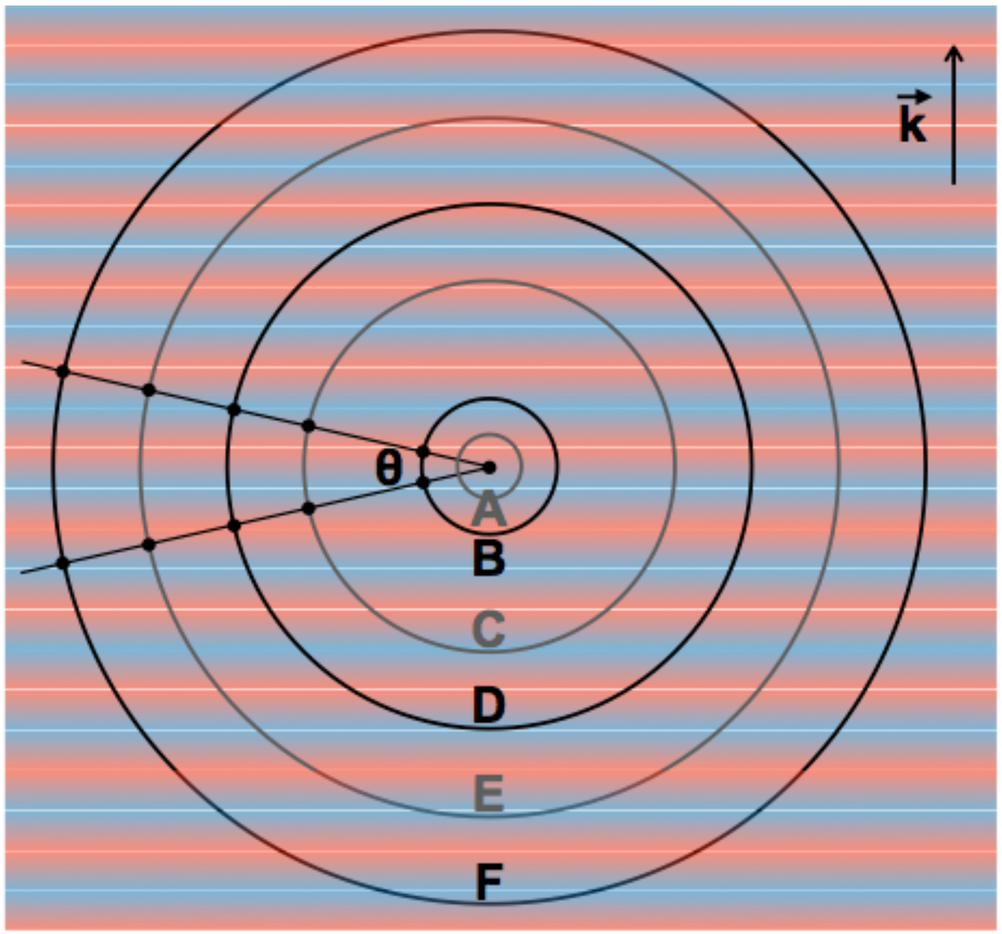
valid both for single mode $\vec{k}$ or transfer function with k

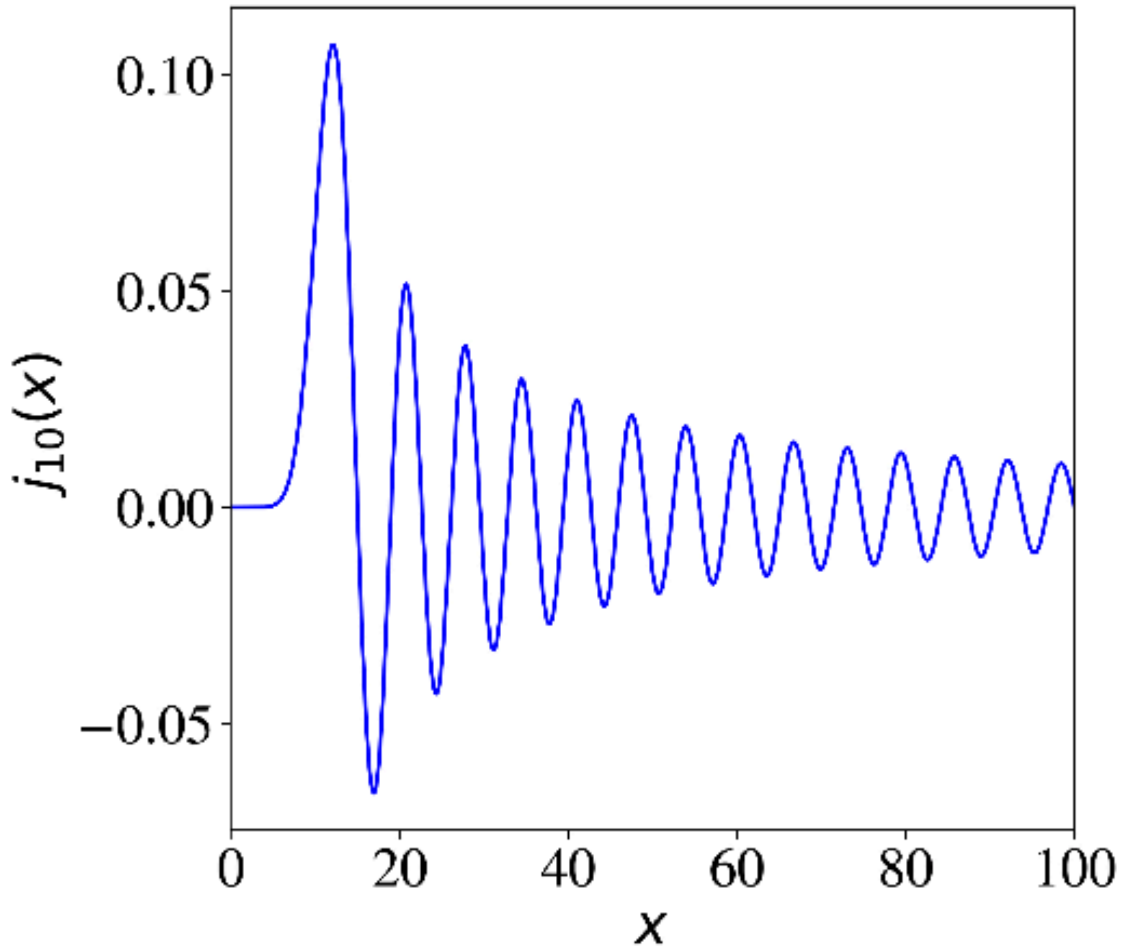
structure: $\int d\eta \; f(\eta) \; A(\eta, \vec{k}) \; j_\ell(k(\eta_0 - \eta))$

“Physical effects relevant at times described by $f(\eta)$ imprint CMB photon anisotropies described in Fourier space by $A(\eta, \vec{k})$, that project to multipole space according to $j_\ell(k(\eta_0 - \eta))$”

Angular projection of Fourier modes

Role of $j_\ell(k(\eta_0 - \eta))$?





Main contribution: $\theta = \dfrac{\pi}{l} = \dfrac{\lambda/2}{d_\mathrm{a}} = \dfrac{a(\eta)\,\pi/k}{a(\eta)\,(\eta_0 - \eta)} \quad \Leftrightarrow \quad l = k(\eta_0 - \eta)$

Other contributions: harmonics

Sachs-Wolfe term

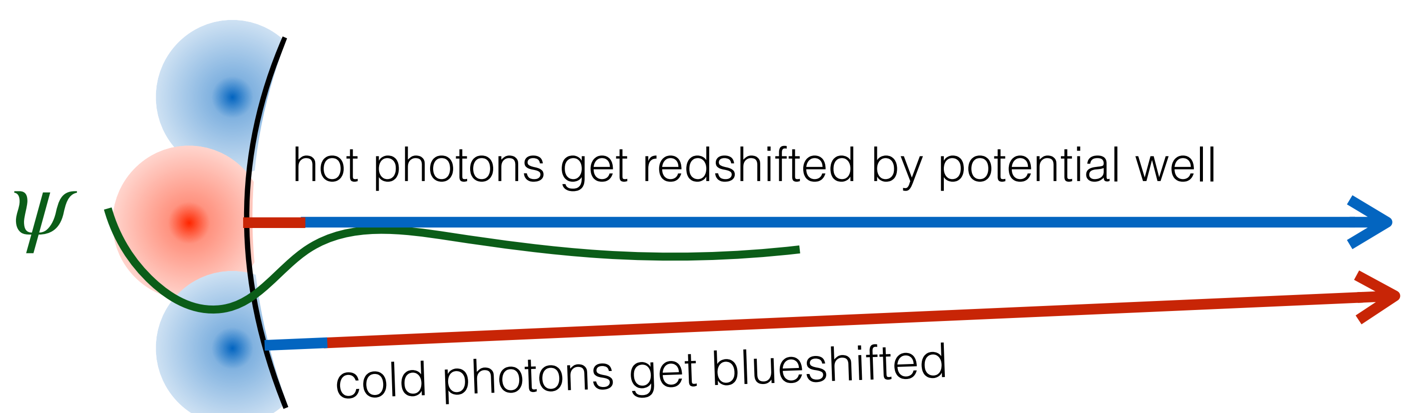
$$\Theta_l(\eta_0,\vec{k}) = \int_{\eta_{\rm ini}}^{\eta_0} d\eta \, \{ g \, (\Theta_0 + \psi) \; j_l(k(\eta_0 - \eta))$$
$$+ \, g \, k^{-1} \theta_{\rm b} \; j_l'(k(\eta_0 - \eta))$$
$$+ \, e^{-\tau} (\phi' + \psi') \, j_l(k(\eta_0 - \eta)) \}$$

Neglecting reionization: $g(\eta)$ very peaked at η_{dec}

$\Rightarrow$ effect takes place only on last scattering sphere

$\Rightarrow$ mode k project to $\ell = k(\eta_0 - \eta_{dec})$

$\Theta_0(\eta_{dec}, \vec{k}) + \psi(\eta_{dec}, \vec{k})$ = intrinsic fluctuation + gravitational Doppler shift



(super-Hubble modes with adiabatic IC: $\psi = -2\Theta_0$, Sachs-Wolfe effect wins, negative picture of last scattering sphere !)