Summary of Lecture 2

Adiabatic initial conditions

⇒ for any perturbation at any time:

$$\forall A, \qquad A(\eta, \vec{k}) = A(\eta, k)\, \mathscr{R}(\eta_i, \vec{k})$$

$$\langle A(\eta, \vec{k})\, A^*(\eta, \vec{k}') \rangle = A(\eta, k)\, A^*(\eta, k')\, \langle \mathscr{R}(\eta_\mathrm{i}, \vec{k})\, \mathscr{R}^*(\eta_\mathrm{i}, \vec{k}') \rangle$$

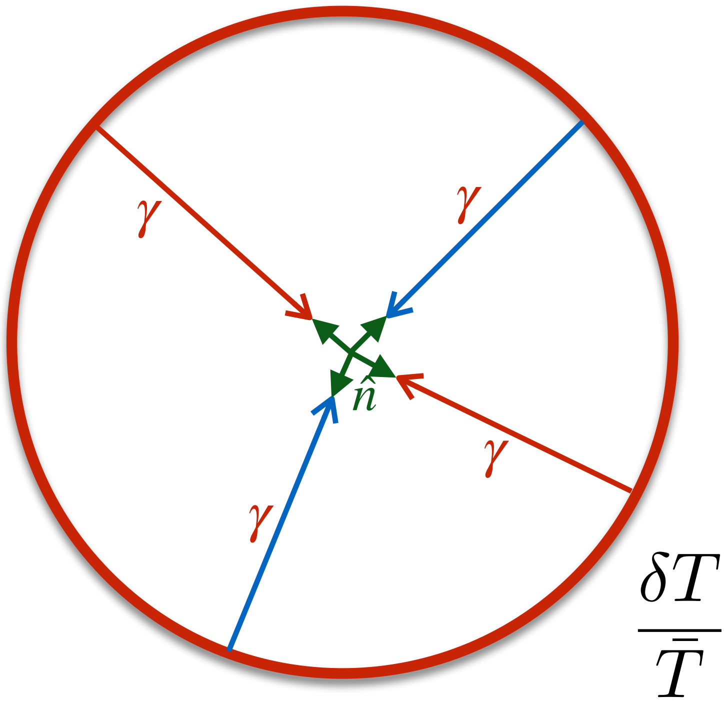
$$= \boxed{|A(\eta, k)|^2 P_{\mathscr{R}}(k)} \quad \delta_D(\vec{k} - \vec{k}')$$

transfer function of A

power spectrum $P_A(\eta, k)$ of A at η

primordial curvature spectrum

Summary of Lecture 2



$$\frac{\delta T}{\bar{T}}(\hat{n}) = \Theta(\eta_0,\vec{o},-\hat{n}) = \sum_{lm} a_{lm} Y_{lm}(\hat{n})$$

CMB spectrum

$$C_l = \langle a_{lm} a^*_{lm} \rangle = \frac{2}{\pi}\int dk\, k^2 \Theta_l^2(\eta_0, k)\, P_{\mathscr{R}}(k)$$

photon transfer function

primordial spectrum

Summary of Lecture 2

Line-of-sight integral in Fourier space: Zaldarriaga & Harari astro-ph/9504085:

$$\Theta_l(\eta_0,\overset{(\rightarrow)}{k}) = \int_{\eta_{\rm ini}}^{\eta_0} d\eta \, \{ g \, (\Theta_0 + \psi) \; j_l(k(\eta_0 - \eta))$$

$$+ \, g \, k^{-1}\theta_{\rm b} \; j_l'(k(\eta_0 - \eta))$$

$$+ \, e^{-\tau} (\phi' + \psi') \, j_l(k(\eta_0 - \eta)) \}$$

structure: $\int d\eta \; f(\eta) \; A(\eta, \vec{k}) \; j_\ell(k(\eta_0 - \eta))$

"Physical effects relevant at times described by $f(\eta)$ imprint CMB photon anisotropies described in Fourier space by $A(\eta, \vec{k})$, that project to multipole space according to $j_\ell(k(\eta_0 - \eta))$"

Sachs-Wolfe term

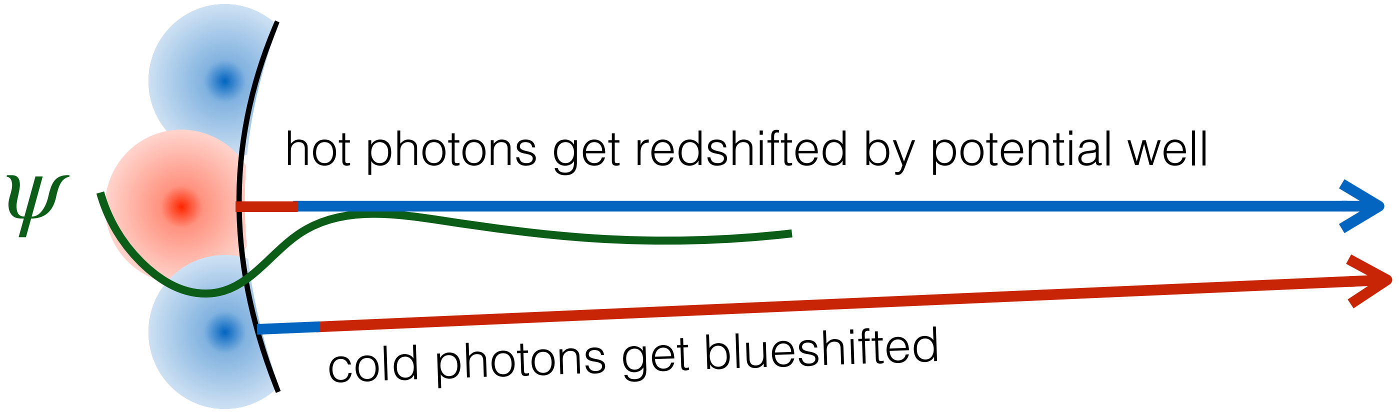
$$\begin{aligned}\Theta_l(\eta_0,\vec{k}) = \int_{\eta_{\rm ini}}^{\eta_0} d\eta\, \{ & g\,(\Theta_0+\psi)\; j_l(k(\eta_0-\eta)) \\ & + g\,k^{-1}\theta_{\rm b}\; j_l'(k(\eta_0-\eta)) \\ & + e^{-\tau}(\phi'+\psi')\, j_l(k(\eta_0-\eta))\}\end{aligned}$$

Neglecting reionization: $g(\eta)$ very peaked at $\eta_{\rm dec}$

⇒ effect takes place only on last scattering sphere

⇒ mode k project to $\ell = k(\eta_0 - \eta_{\rm dec})$

$\Theta_0(\eta_{\rm dec}, \vec{k}) + \psi(\eta_{\rm dec}, \vec{k})$ = intrinsic fluctuation + gravitational Doppler shift



(super-Hubble modes with adiabatic IC: $\psi = -2\Theta_0$, Sachs-Wolfe effect wins, negative picture of last scattering sphere !)

Doppler term

$$\Theta_l(\eta_0,\vec{k}) = \int_{\eta_{\rm ini}}^{\eta_0} d\eta \, \{ g\,(\Theta_0+\psi)\; j_l(k(\eta_0-\eta))$$

$$+ \, g \, k^{-1}\theta_{\rm b} \; j_l'(k(\eta_0-\eta))$$

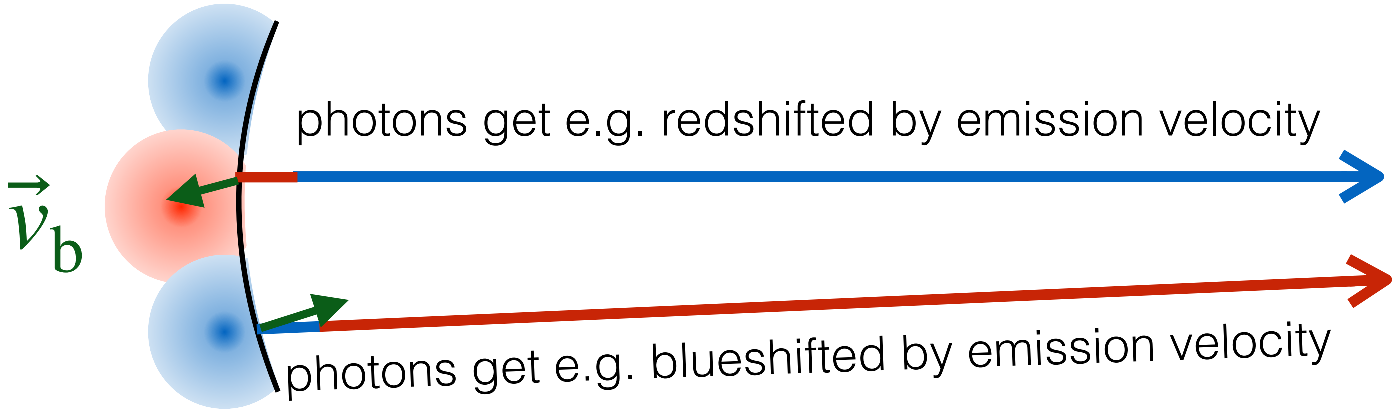
$$+ \; e^{-\tau}(\phi'+\psi')\, j_l(k(\eta_0-\eta)) \}$$

Neglecting reionization: $g(\eta)$ very peaked at $\eta_{\rm dec}$

$\Rightarrow$ effect takes place only on last scattering sphere

$\Rightarrow$ mode k project to $\ell = k(\eta_0 - \eta_{\rm dec})$

$\hat{n}\cdot\vec{v}_{\rm b}^{\,\rm scalar} \rightarrow k^{-1}\theta_{\rm b}$ = velocity Doppler shift (j_ℓ' from a gradient)



Integrated Sachs-Wolfe (ISW) term

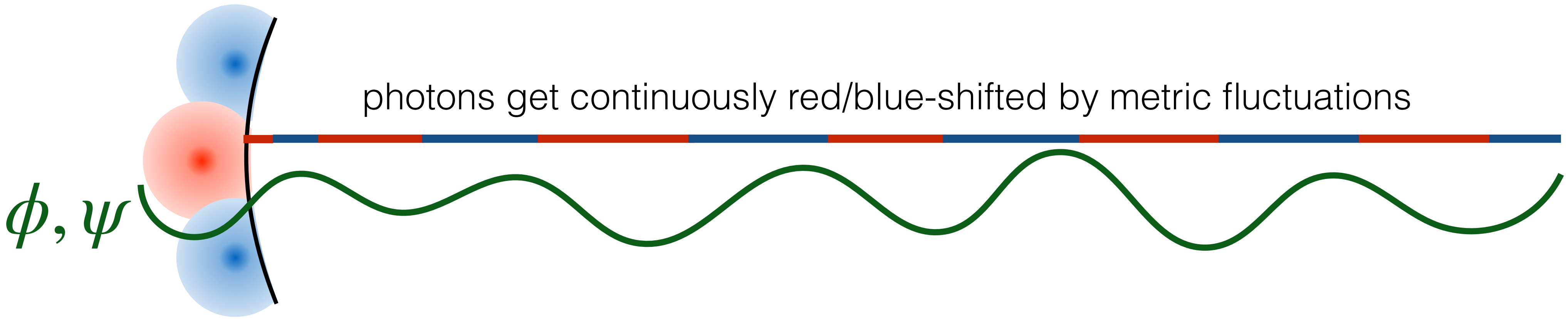
$$\Theta_l(\eta_0,\vec{k}) = \; \dots \; + e^{-\tau} (\phi' + \psi') \, j_l(k(\eta_0 - \eta))$$

Neglecting reionization: $e^{-\tau}$ negligible before $\eta_{\rm dec}$, $\simeq 1$after

$\Rightarrow$ effect takes place at all times $\eta > \eta_{\rm dec}$ along each line of sight

$\Rightarrow$ mode k projects from each sphere to $\ell = k(\eta_0 - \eta)$

$\partial_\eta \{ \phi(\eta,\vec{k}) + \psi(\eta,\vec{k}) \}$ comes from dilation + gravitational Doppler effects



- ϕ, ψ static: no dilation, gravitational Doppler effect is conservative: only $(\psi_{\rm dec} - \psi_{\rm obs})$
- ϕ, ψ time-dependent: net effect (e.g. net redshift when crosses deepening potential wells)

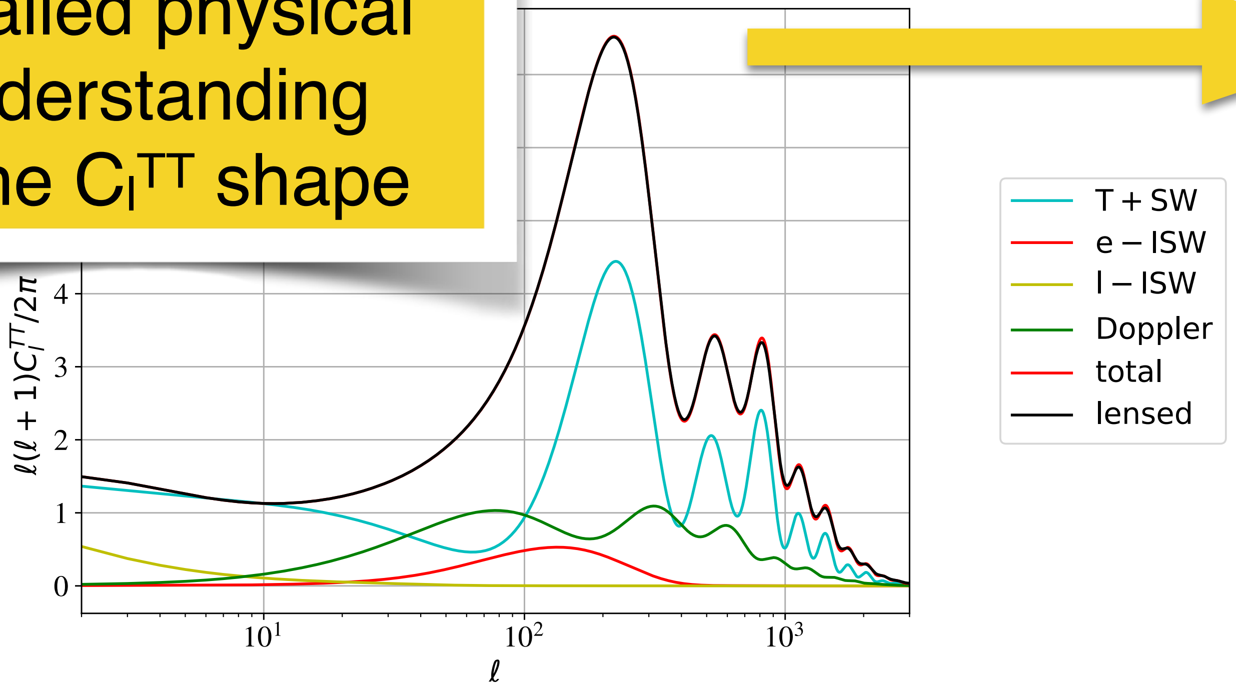
Summary

Final goal: compute $C_\ell = \langle a_{lm} a^*_{lm} \rangle = \frac{2}{\pi} \int dk\, k^2 \Theta_\ell^2(\eta_0, k) \;\; P_{\mathscr{R}}(k)$

with transfer functions

$$\Theta_l(\eta_0,k) = \int_{\eta_\mathrm{ini}}^{\eta_0} d\eta \, \{ g \, (\Theta_0 + \psi) \; j_l(k(\eta_0 - \eta))$$
$$+ \, g \, k^{-1} \theta_\mathrm{b} \; j_l'(k(\eta_0 - \eta))$$
$$+ \; e^{-\tau} (\phi' + \psi') \, j_l(k(\eta_0 - \eta)) \}$$

Detailed physical understanding of the C_l^{TT} shape



behaviour of

$$\Theta_0(\eta_\mathrm{dec}, k)$$
$$\theta_\mathrm{b}(\eta_\mathrm{dec}, k)$$
$$\psi(\eta \geq \eta_\mathrm{dec}, k) \simeq \phi$$

Tight-Coupling Approximation (TCA)

When $\Gamma_\gamma \gg \dfrac{a'}{a}$:
tightly-coupled baryon-photon fluid:
$$\begin{cases} \Theta_0 = \frac{1}{4}\delta_\gamma = \frac{1}{3}\delta_b \\ 3k\Theta_1 = \theta_\gamma = \theta_b \\ \Theta_{l\geq 2} = 0 \end{cases}$$

$\Theta_0 = \frac{1}{4}\delta_\gamma = \frac{1}{3}\delta_b$ ⟶ from thermal equilibrium

$3k\Theta_1 = \theta_\gamma = \theta_b$, $\Theta_{l\geq 2} = 0$ ⟶ from efficient Thomson scattering

⇒ photon Boltzmann hierarchy + baryon fluid equations —> single TCA equation:

$$\Theta_0'' + \underbrace{\frac{R}{1+R}\frac{a'}{a}\Theta_0'}_{\text{baryon damping}} + \underbrace{k^2 c_{\mathrm{s}}^2 \Theta_0}_{\text{pressure force}} = \underbrace{-\frac{k^2}{3}\psi}_{\text{gravity force}} + \underbrace{\frac{R}{1+R}\frac{a'}{a}\phi'}_{\text{local baryon damping}} + \underbrace{\phi''}_{\text{dilation}}$$

Squared sound speed / baryon-to-photon ratio: $c_{\mathrm{s}}^2 = \dfrac{1}{3(1+R)}$, $R \equiv \dfrac{3\bar{\rho}_{\mathrm{b}}}{4\bar{\rho}_\gamma} \propto a$

Tight-coupling equation

$$\Theta_0'' + \underset{\text{baryon damping}}{\frac{R}{1+R}\frac{a'}{a}\Theta_0'} + \underset{\text{pressure force}}{k^2 c_\mathrm{s}^2 \Theta_0} = \underset{\text{gravity force}}{-\frac{k^2}{3}\psi} + \underset{\text{local baryon damping}}{\frac{R}{1+R}\frac{a'}{a}\phi'} + \underset{\text{dilation}}{\phi''}$$

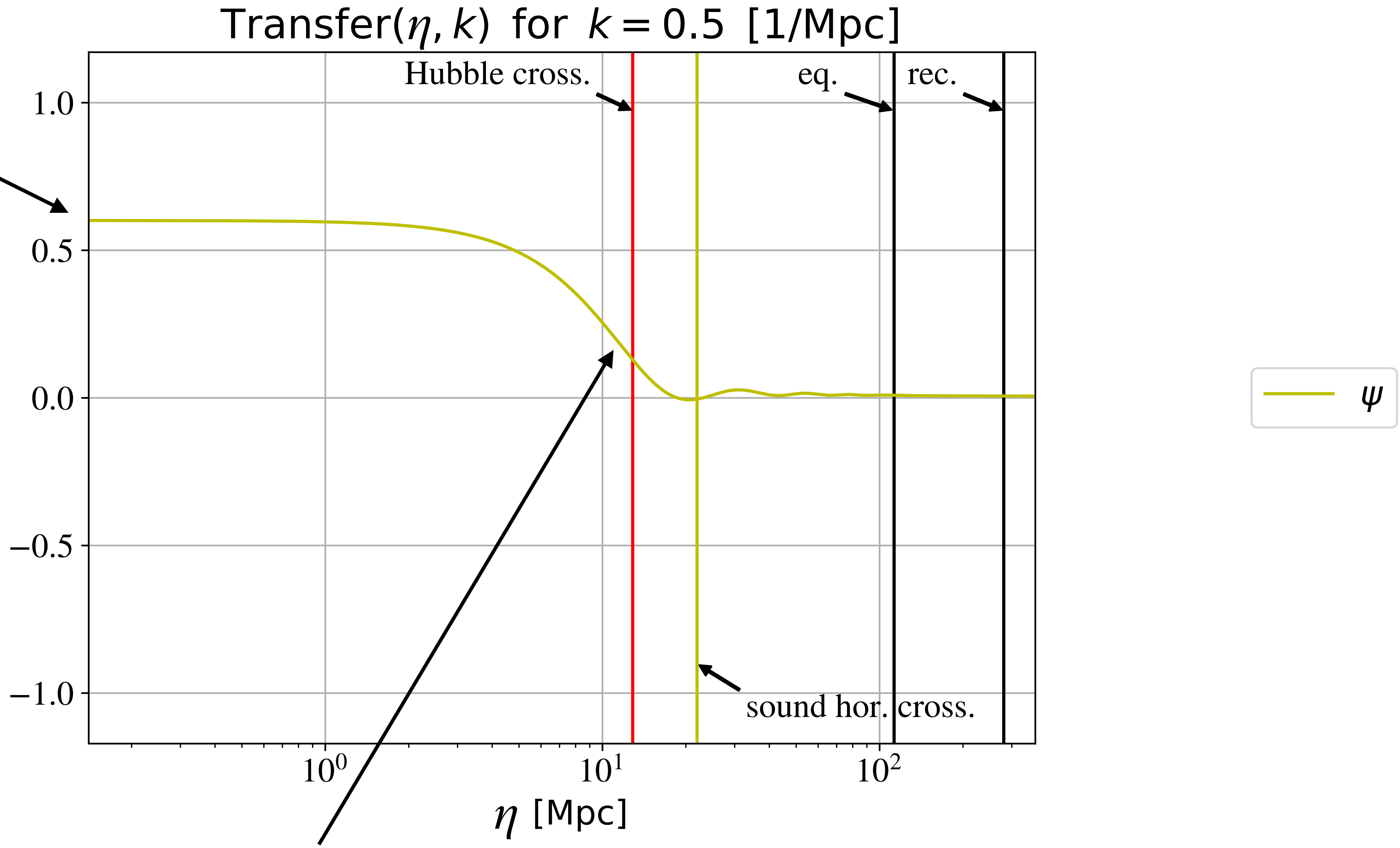
Squared sound speed / baryon-to-photon ratio: $c_\mathrm{s}^2 = \dfrac{1}{3(1+R)}$, $R \equiv \dfrac{4\bar{\rho}_\mathrm{b}}{3\bar{\rho}_\gamma} \propto a$

Equilibrium point neglecting metric time derivatives: $\Theta_0^{\mathrm{equi.}} = -\dfrac{1}{3c_\mathrm{s}^2}\psi = -(1+R)\psi$

WKB TCA solution “ “ “ : $\Theta_0 = A(1+R)^{-1/4}\cos\left(k \int c_\mathrm{s}(\eta)d\eta\right) - (1+R)\psi$

Very good approximation up to gravity boost + (Silk) damping/diffusion effects

Evolution for one mode with given k



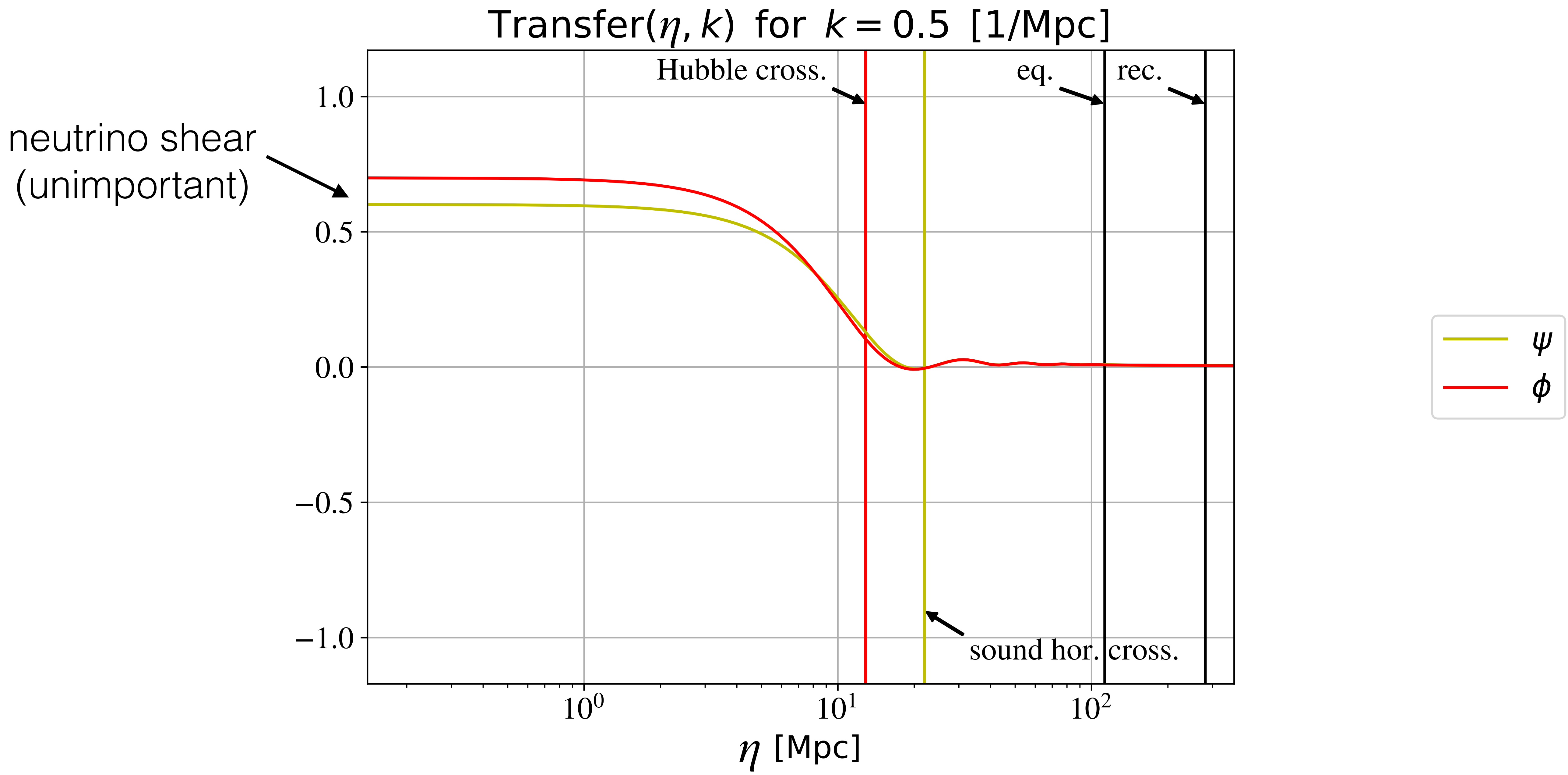
Adiabatic IC
($\mathcal{R}$=1, ψ= 2/3)

Metric damped near Hubble crossing during RD

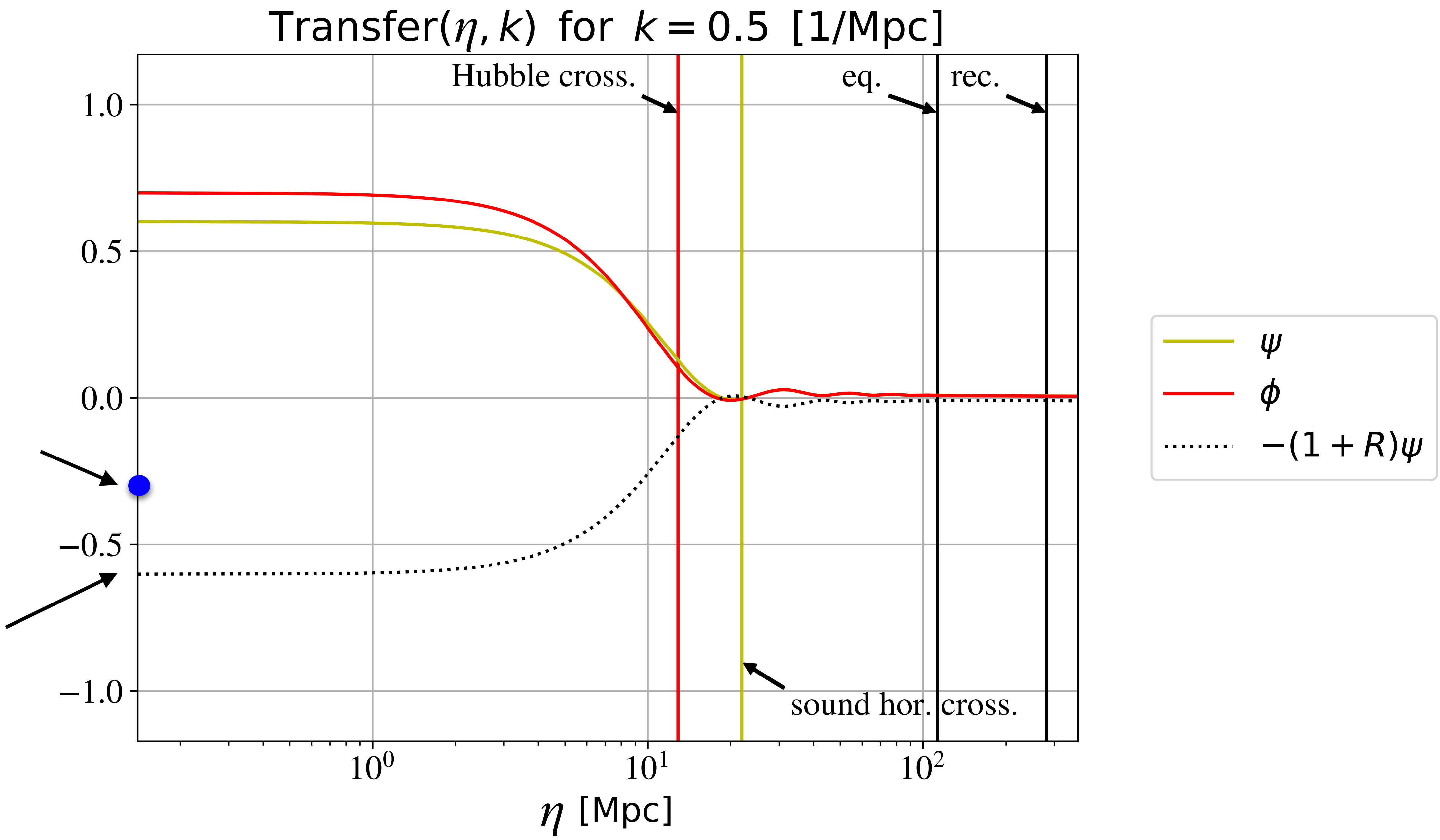
—> photon pressure, Poisson: $-k^2\phi = 4\pi G\, a^2\, \delta\rho_r \propto a^2 \rho_r\, \delta_r \sim a^{2-4+0} \sim a^{-2}$

—> very different from MD: $-k^2\phi = 4\pi G\, a^2\, \delta\rho_m \propto a^2 \rho_m\, \delta_m \sim a^{2-3+1} \sim \mathrm{constant}$

Evolution for one mode



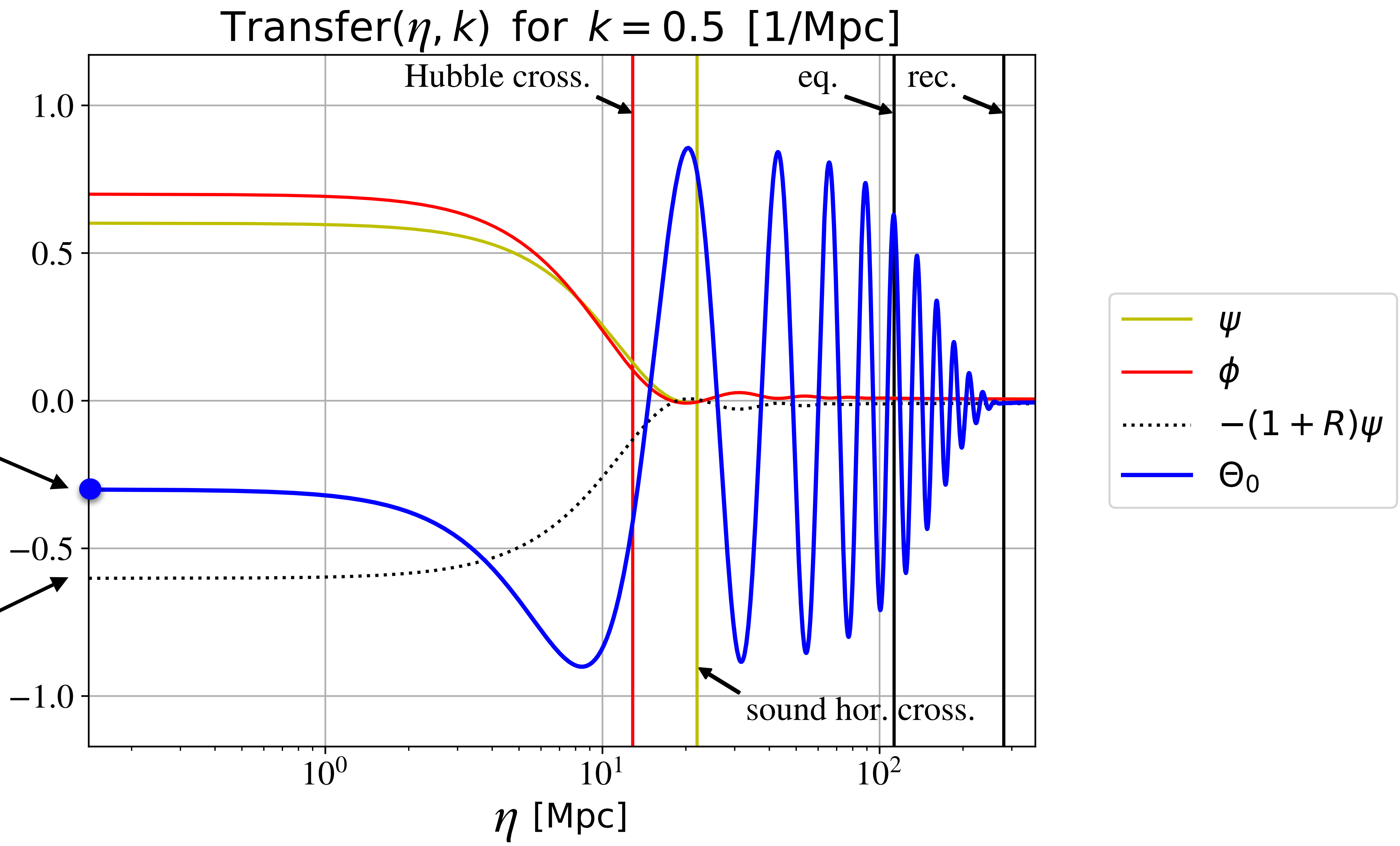
Evolution for one mode



photon IC
Θ_0 = -1/3

photon
equilibrium
Θ_0 = -2/3

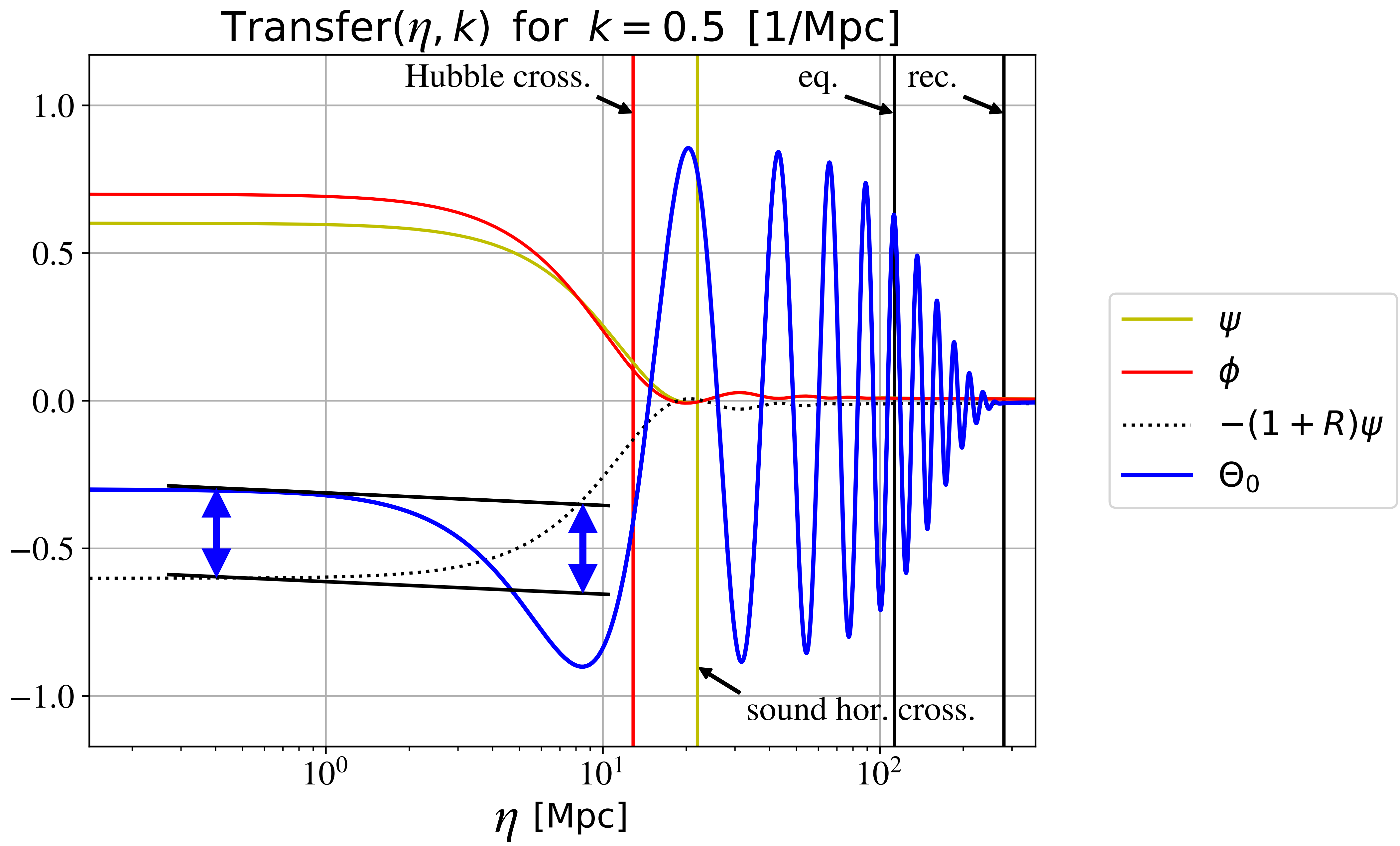
Evolution for one mode



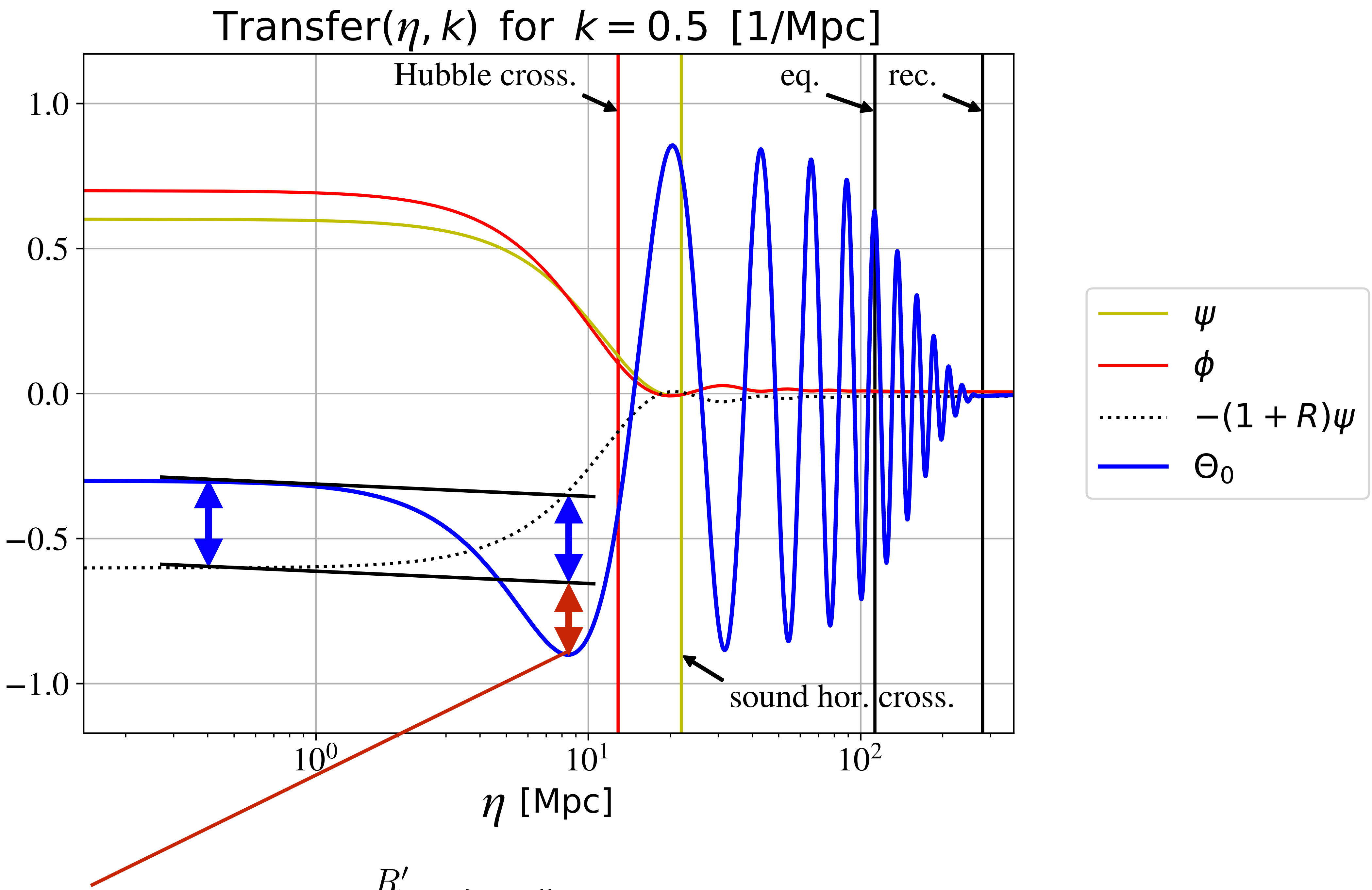
photon IC
Θ_0 = -1/3

photon
equilibrium
Θ_0 = -2/3

Evolution for one mode



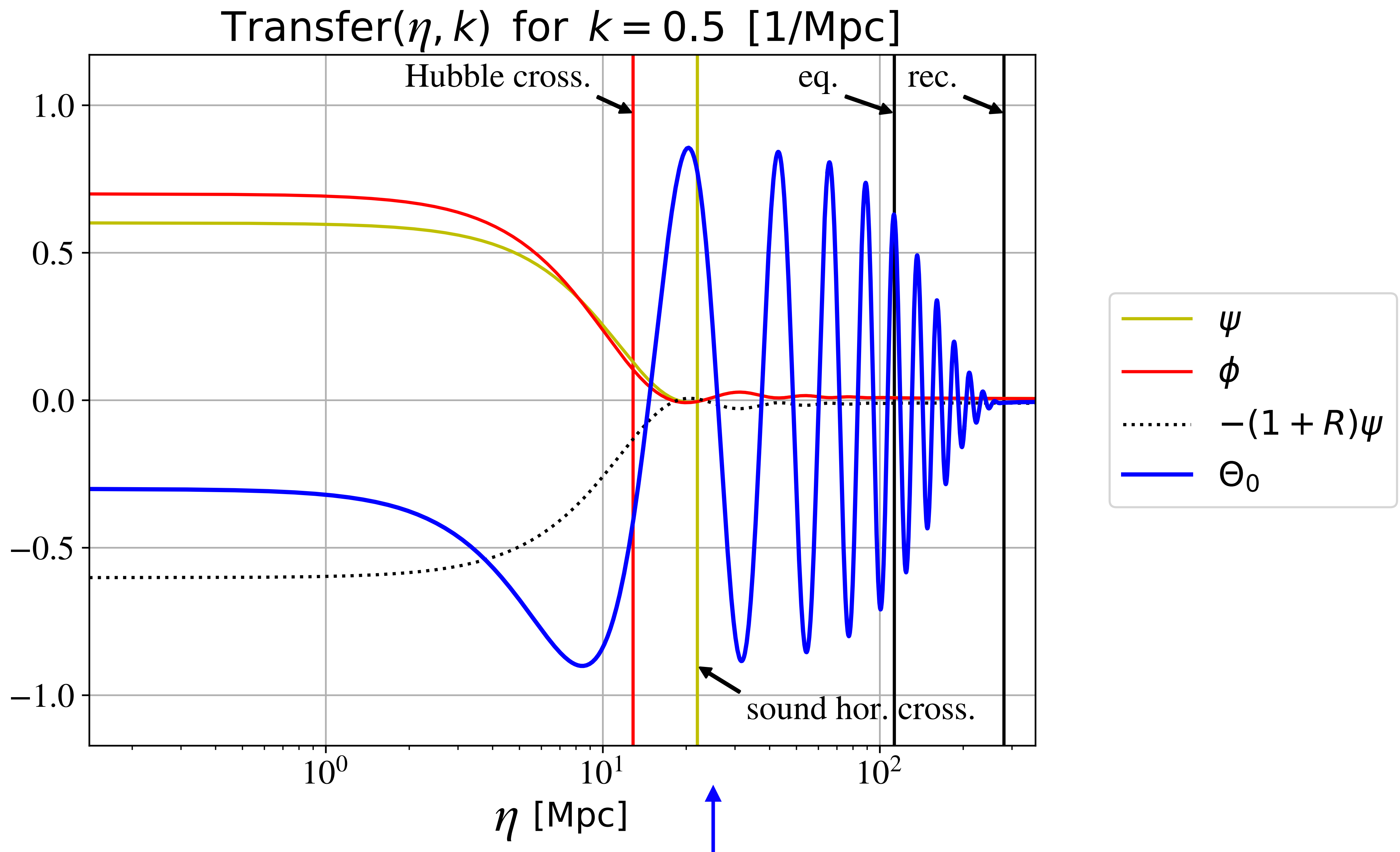
Evolution for one mode



Gravity boost effect from $\frac{R'}{1+R}\phi' + \phi''$

Will be important for effect of neutrinos, DR…

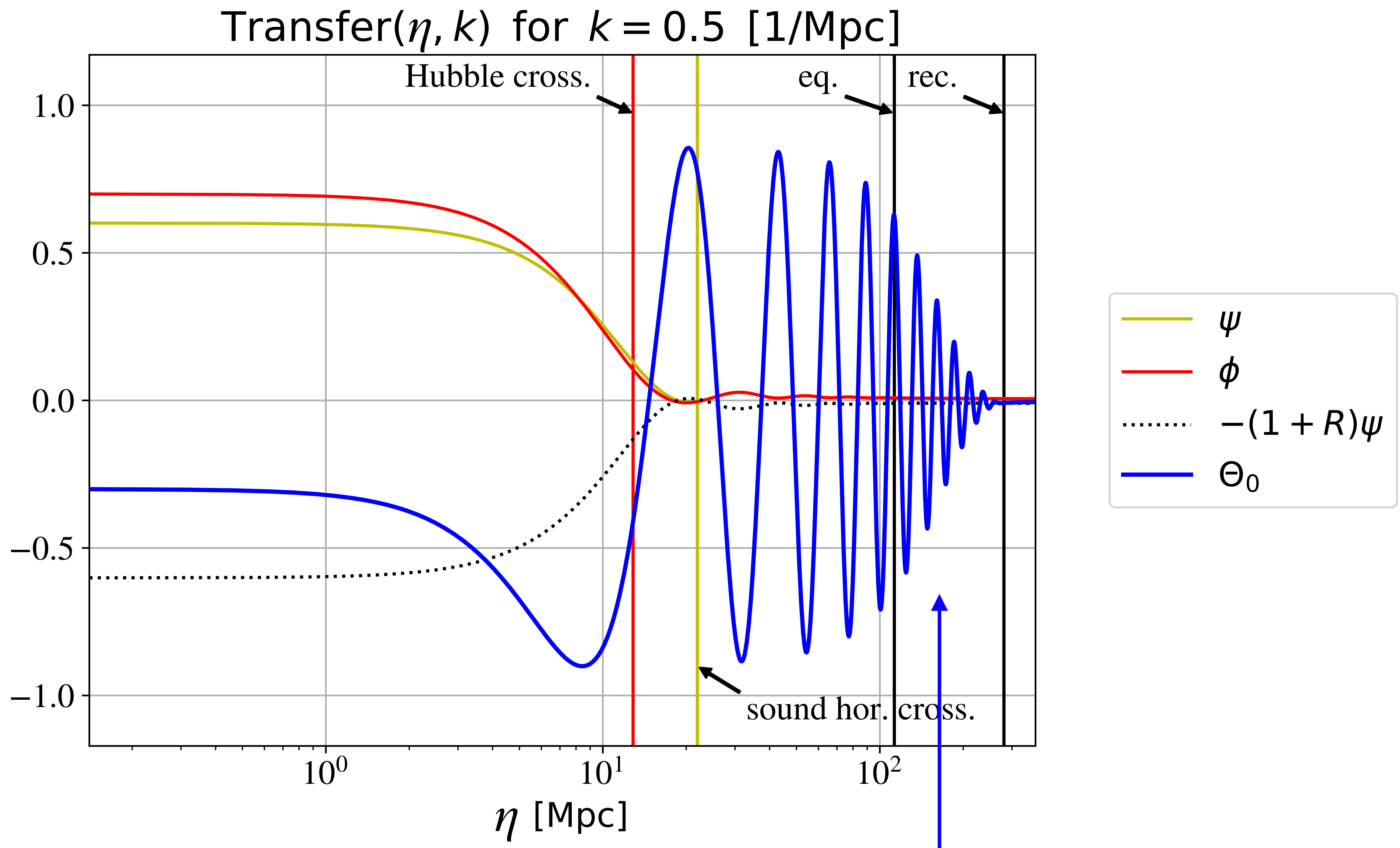
Evolution for one mode



symmetric and stationary oscillation

(deep sub-Hubble, deep DR)

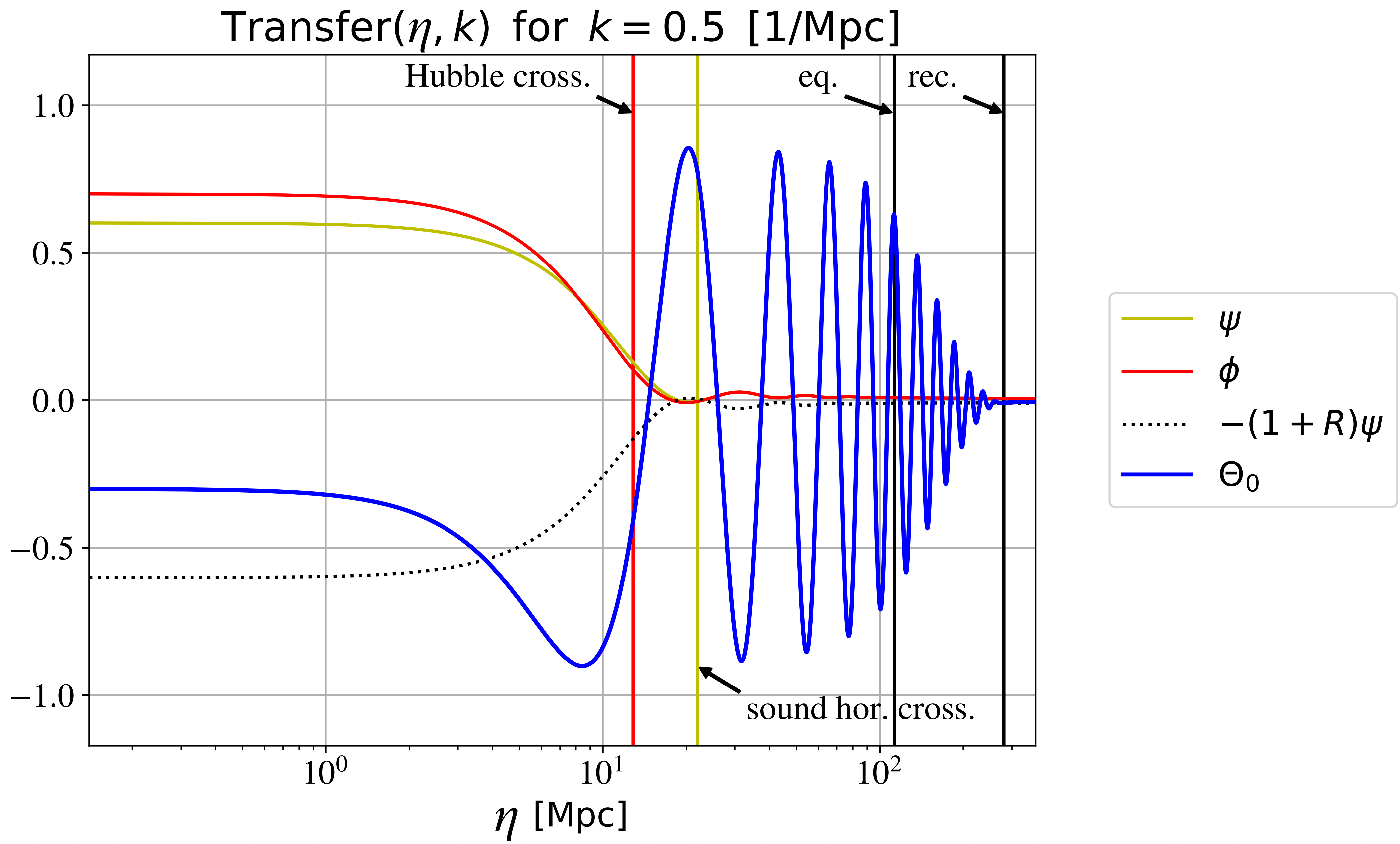
Evolution for one mode



exponentially damped oscillations

(approaching recombination)

Evolution for one mode


Transfer(η, k) for $k = 0.5$ [1/Mpc]

Final goal:

(MZ’s line-of-sight integral)

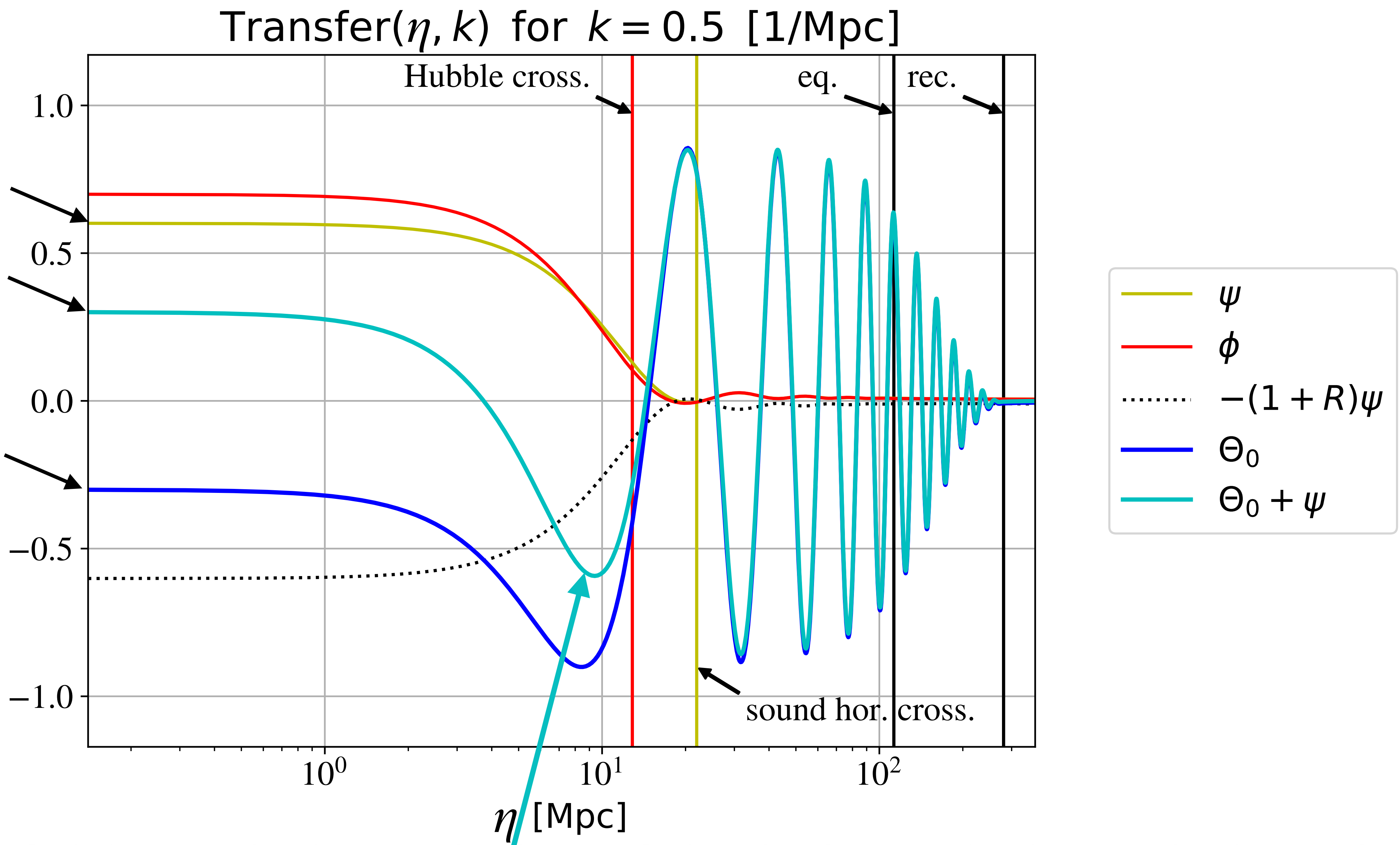
$$\Theta_l(\eta_0,k) = \int_{\eta_{\rm ini}}^{\eta_0} d\eta \left(\underbrace{g\,(\Theta_0 + \psi)}_{\rm SW} + \underbrace{\left(g\,k^{-2}\theta_{\rm b}\right)'}_{\rm Doppler} + \underbrace{e^{-\tau}(\phi' + \psi')}_{\rm ISW} \right) j_l(k(\eta_0 - \eta))$$

Evolution for one mode

Adiabatic IC
($\mathcal{R}$=1, ψ= 2/3)

$\Theta_0 + \psi = 1/3$

photon IC
Θ_0 = -1/3

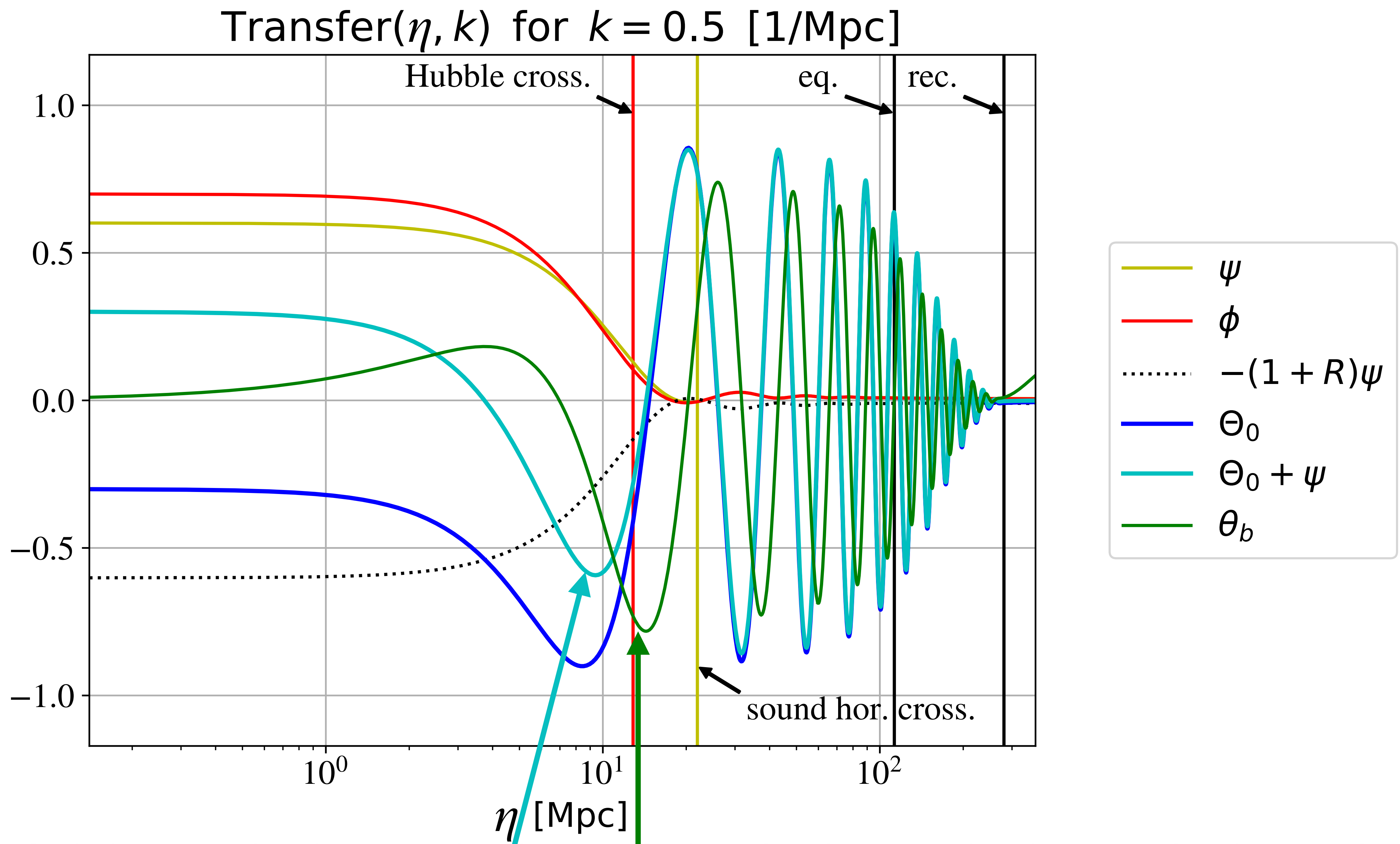


Final goal:

(MZ's line-of-sight integral)

$$\Theta_l(\eta_0,k) = \int_{\eta_{\rm ini}}^{\eta_0} d\eta \left(\underbrace{g\,(\Theta_0+\psi)}_{\rm SW} + \underbrace{\left(g\,k^{-2}\theta_{\rm b}\right)'}_{\rm Doppler} + \underbrace{e^{-\tau}(\phi'+\psi')}_{\rm ISW} \right) j_l(k(\eta_0-\eta))$$

Evolution for one mode



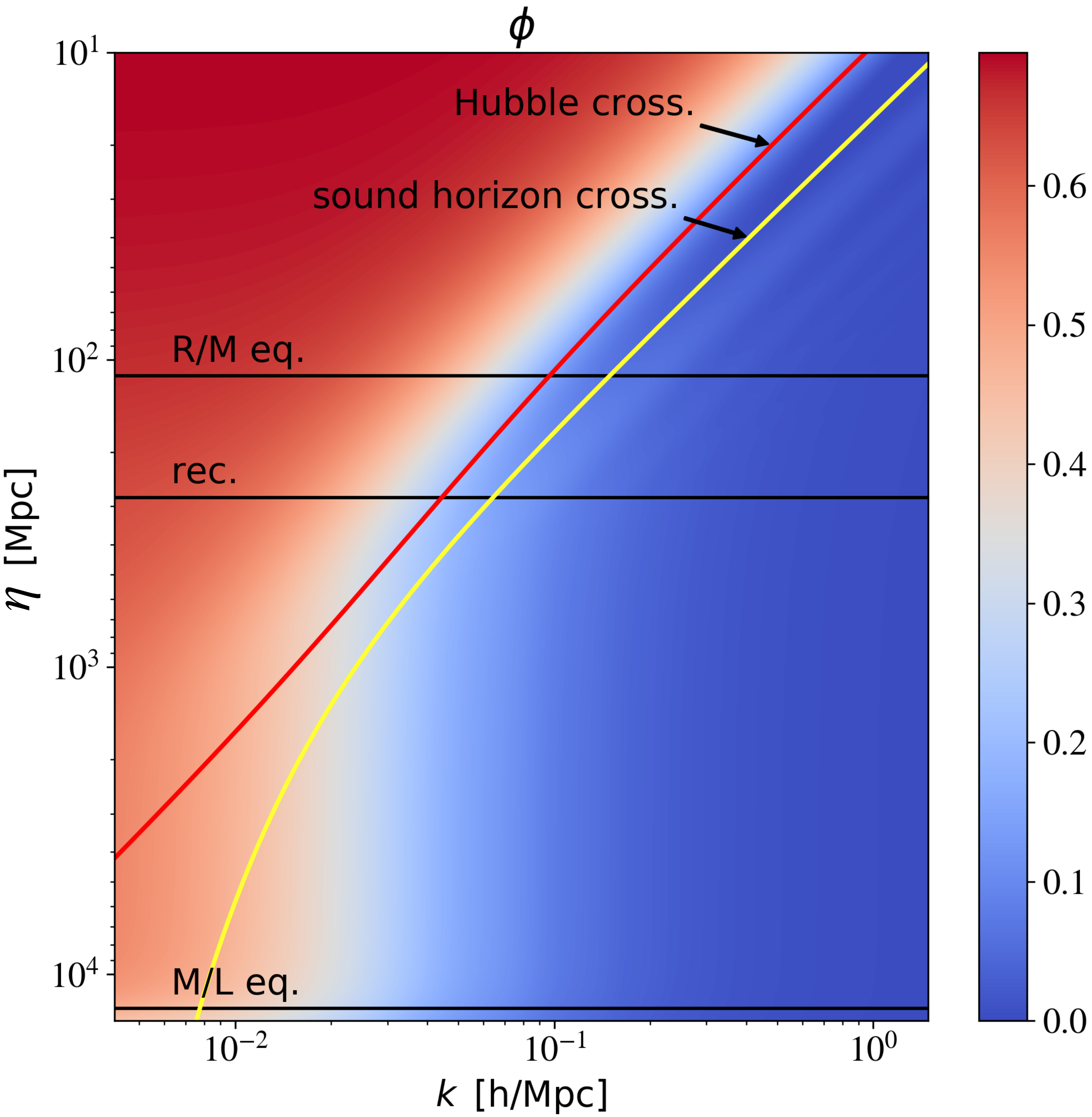
Final goal:

(MZ's line-of-sight integral)

$$\Theta_l(\eta_0,k) = \int_{\eta_{\rm ini}}^{\eta_0} d\eta \left(\underbrace{g\,(\Theta_0+\psi)}_{\rm SW} + \underbrace{\left(g\,k^{-2}\theta_{\rm b}\right)'}_{\rm Doppler} + \underbrace{e^{-\tau}(\phi'+\psi')}_{\rm ISW}\right) j_l(k(\eta_0-\eta))$$

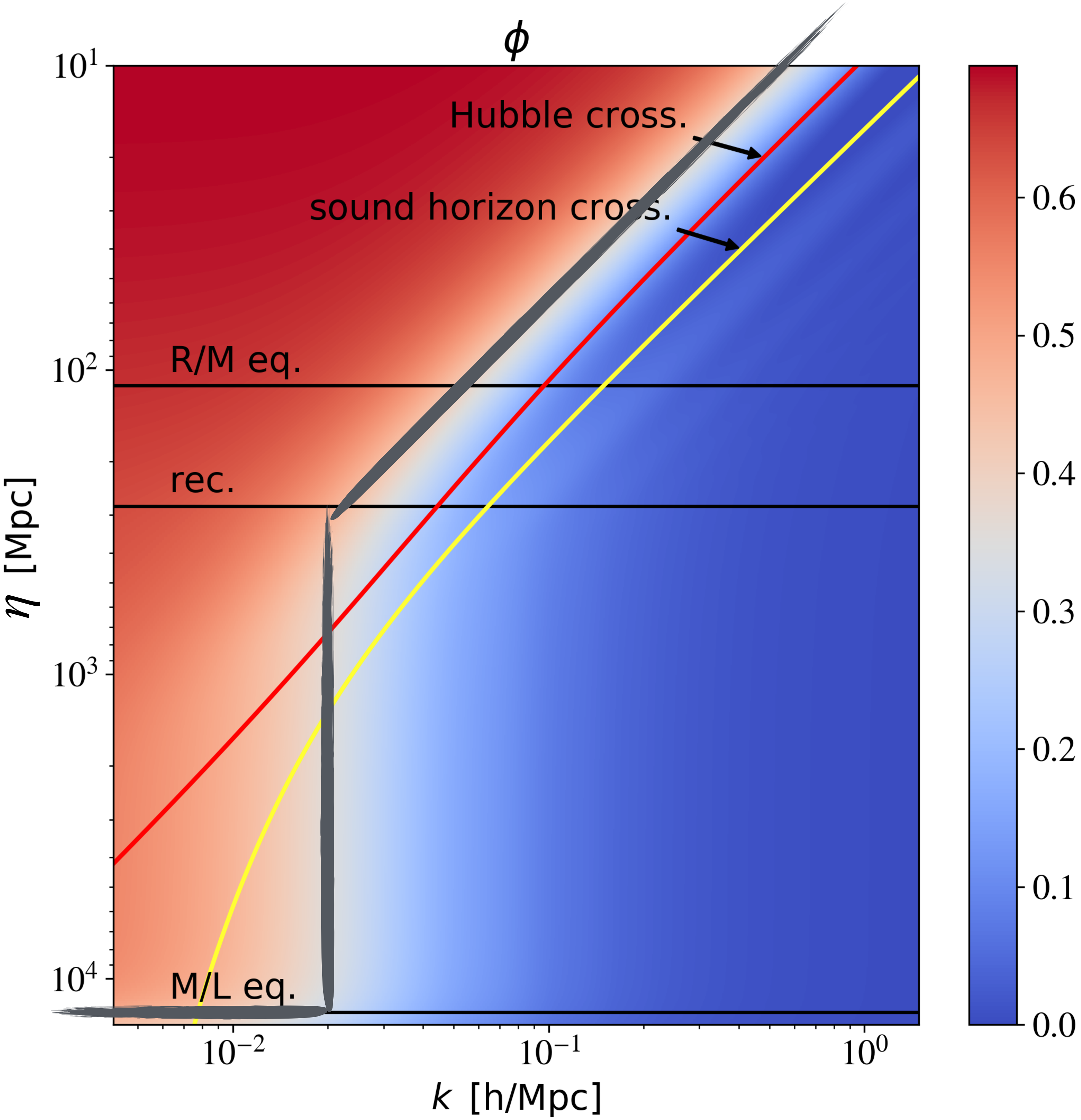
Evolution for all wavenumbers

Metric $\phi(\eta, k)$:



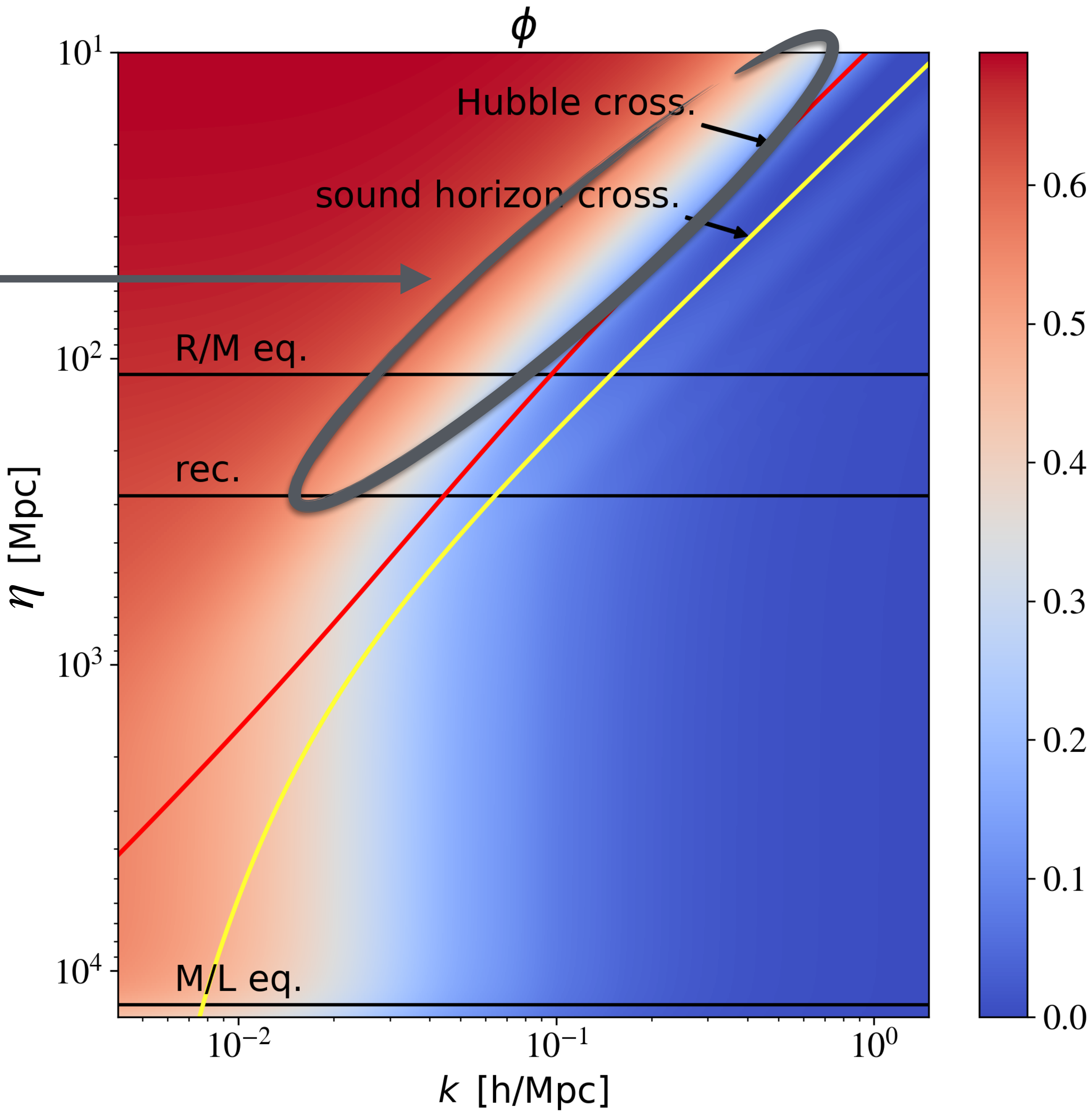
Evolution for all wavenumbers

Metric $\phi(\eta, k)$:

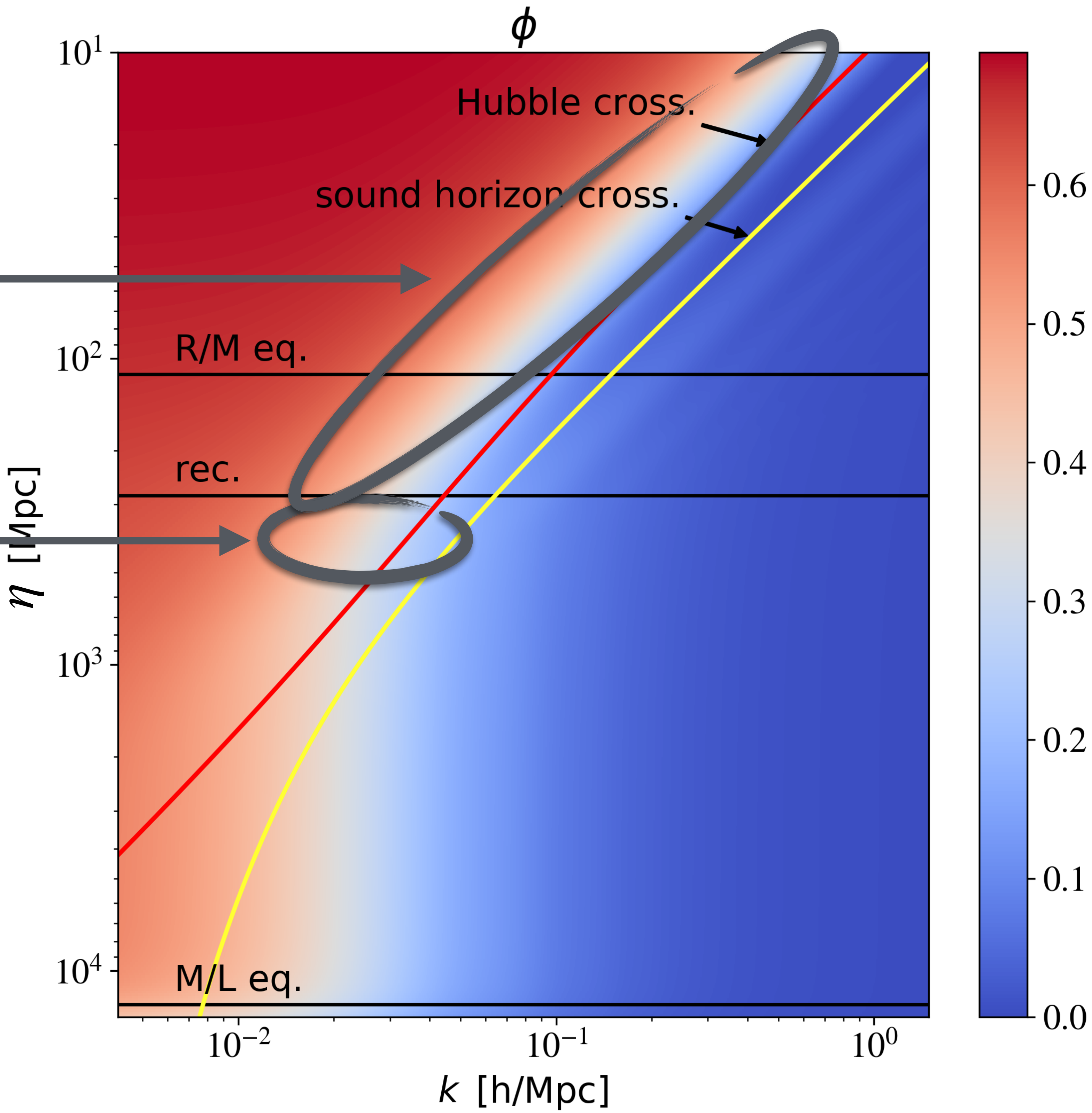


Evolution for all wavenumbers

$$\frac{R'}{1+R}\phi' + \phi'' \Rightarrow \text{gravity boost}$$



Evolution for all wavenumbers



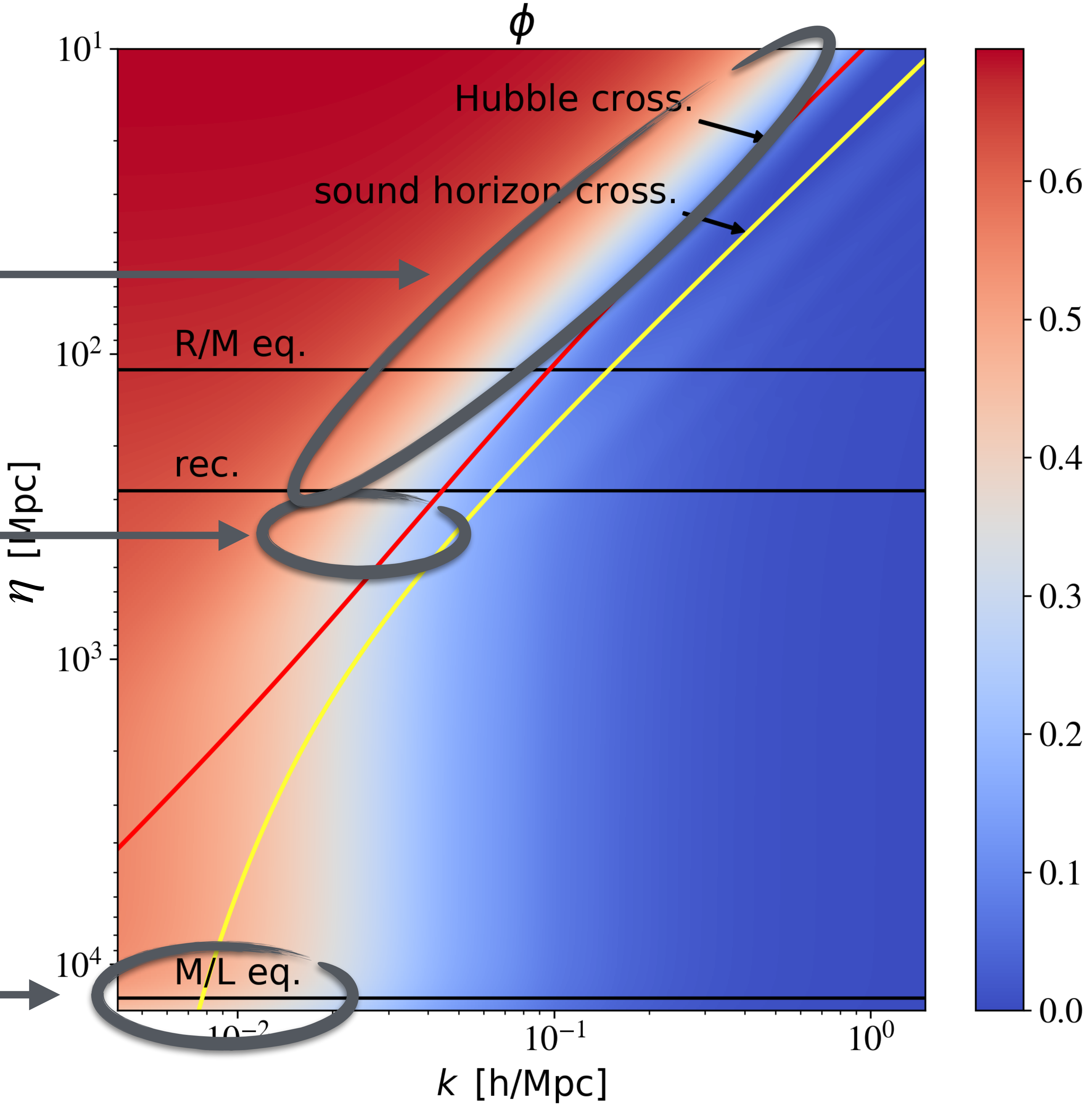
$$\frac{R'}{1+R}\phi' + \phi''$$

gravity boost

$$\underbrace{e^{-\tau}(\phi' + \psi')}_{\text{ISW}}$$

early ISW

Evolution for all wavenumbers

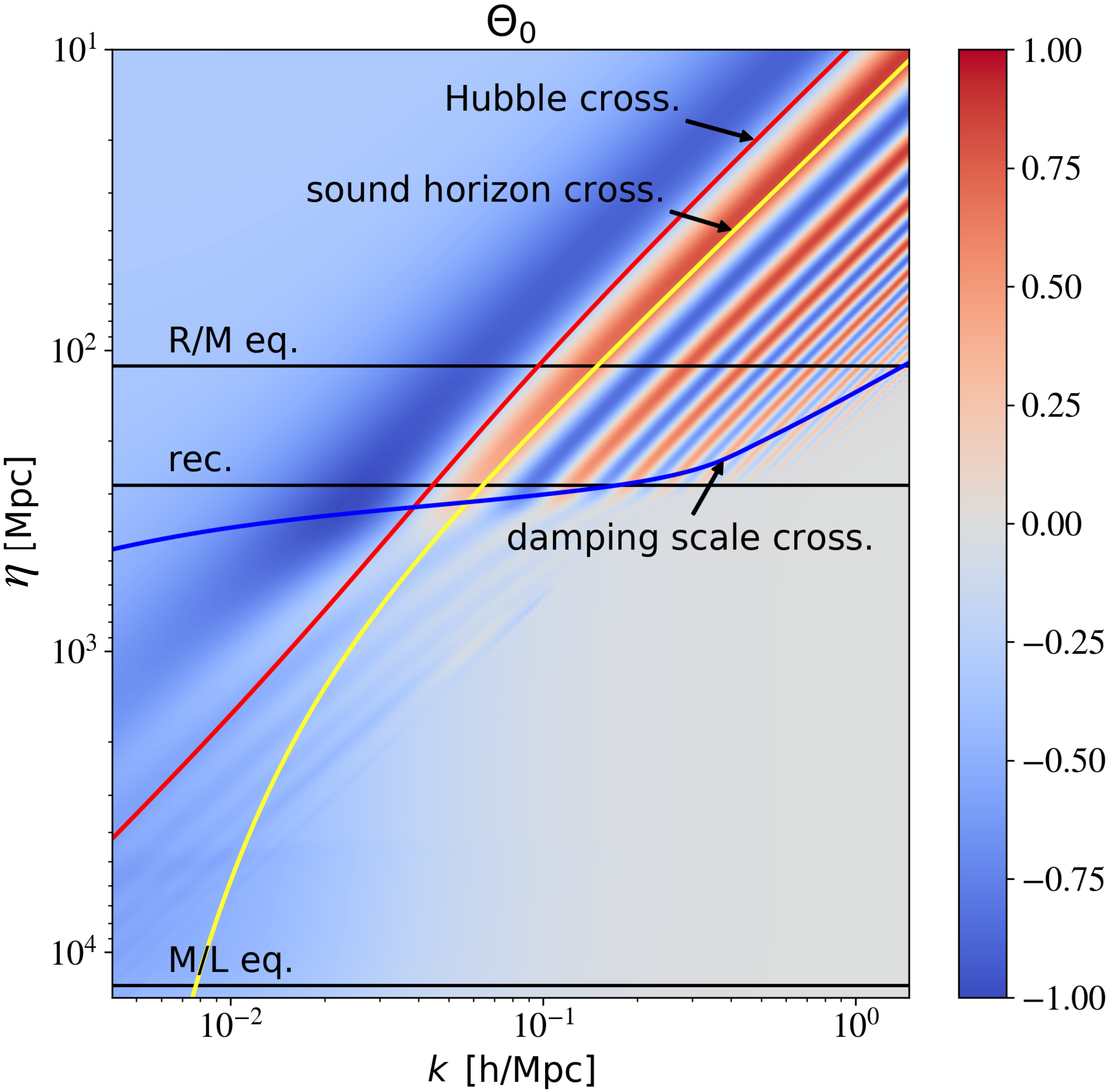


$\frac{R'}{1+R}\phi' + \phi''$ gravity boost

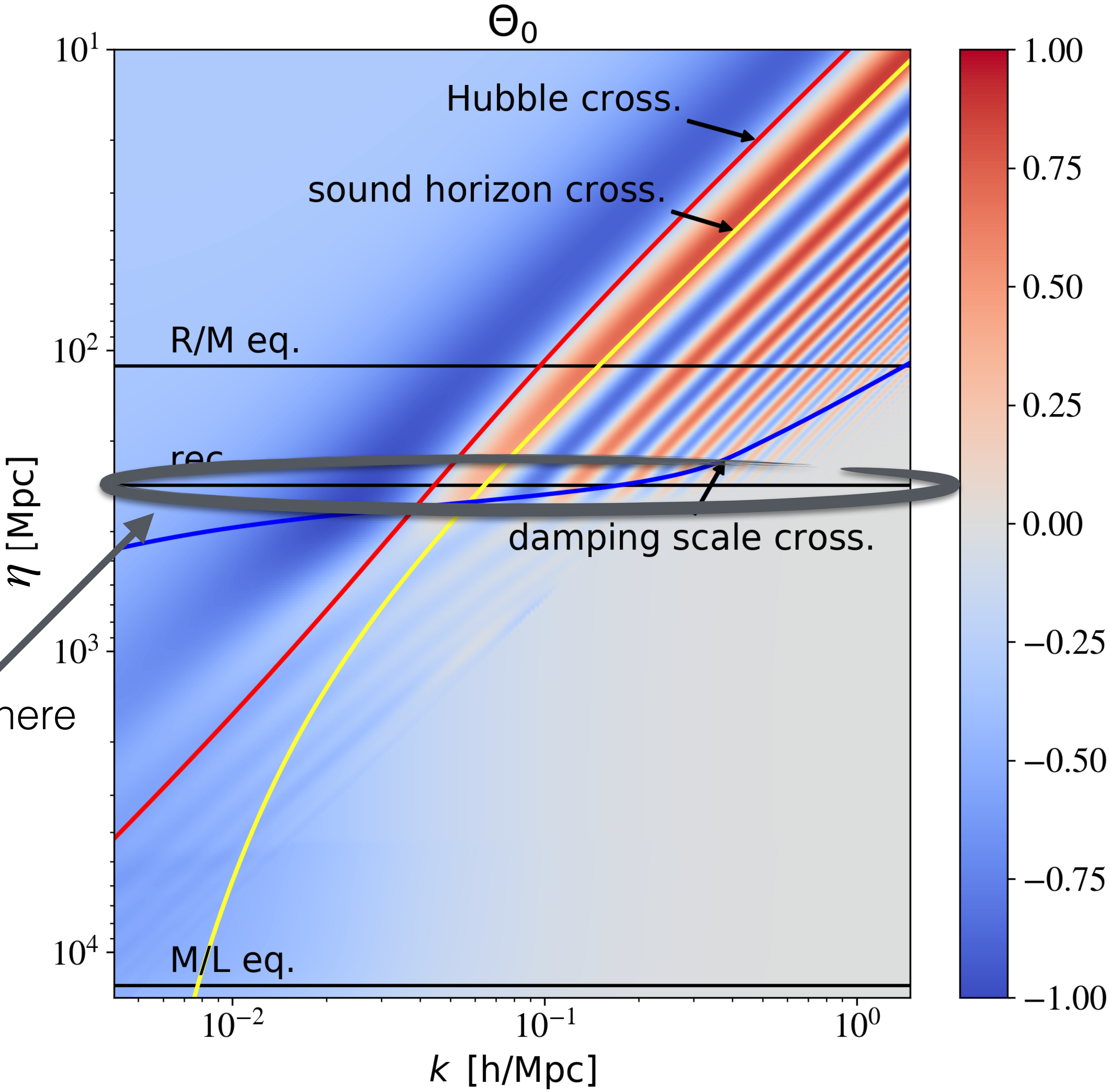
$\underbrace{e^{-\tau}(\phi' + \psi')}_{\text{ISW}}$ early ISW

$\underbrace{e^{-\tau}(\phi' + \psi')}_{\text{ISW}}$ late ISW

Evolution for all wavenumbers

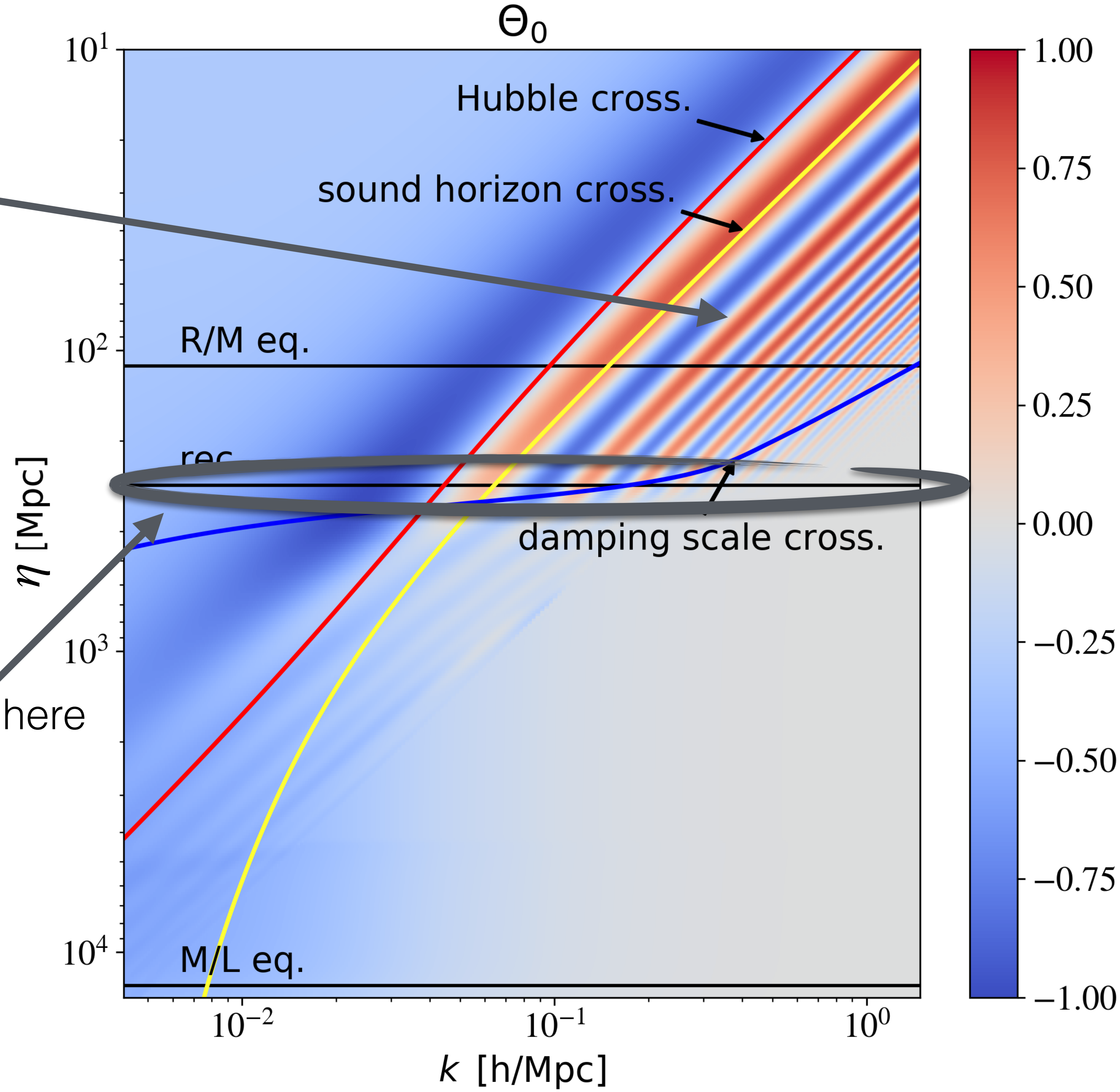


Evolution for all wavenumbers



CMB patterns mainly come from here

Evolution for all wavenumbers

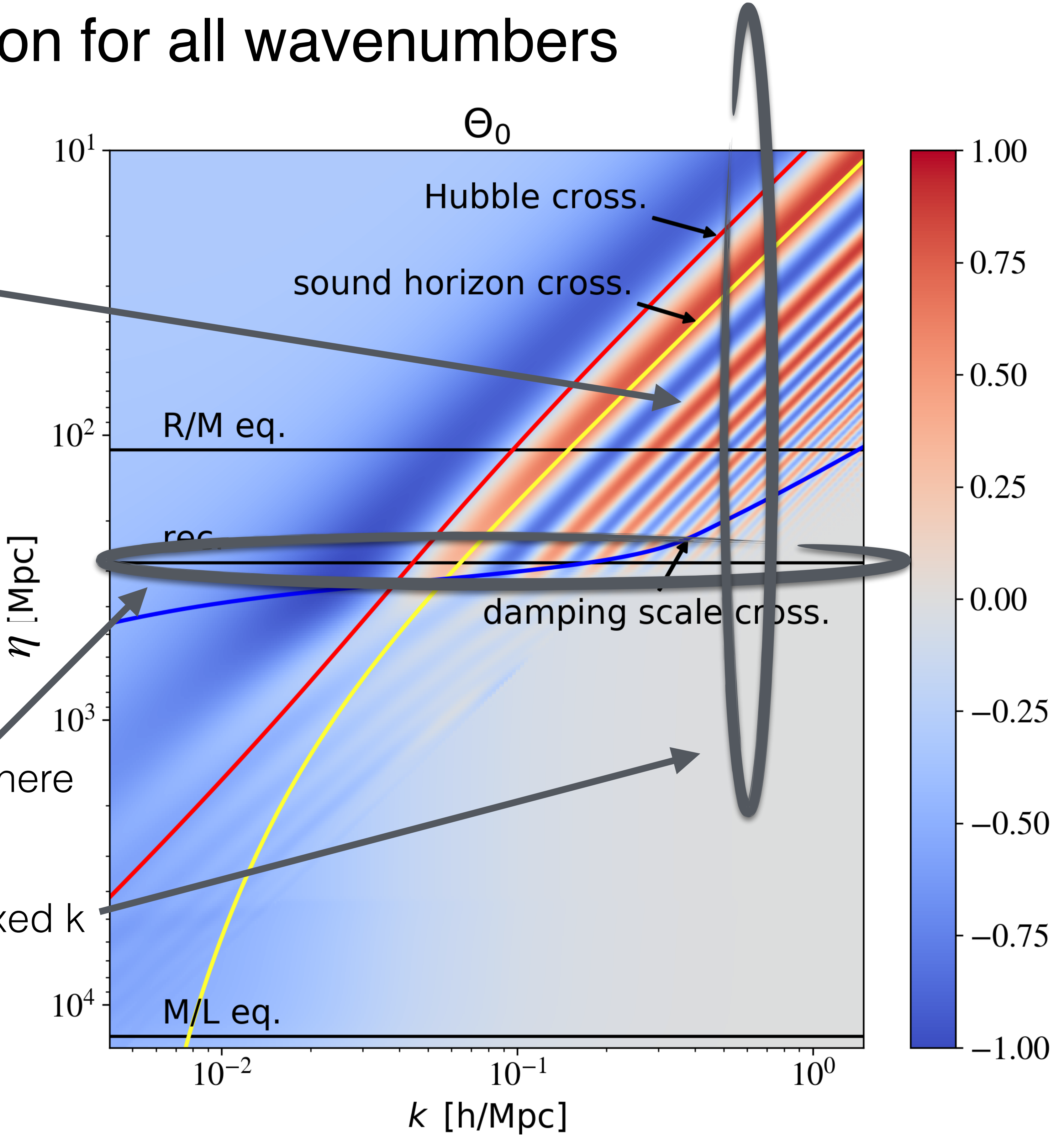


45° symmetry comes from

$$\cos(kc_s\eta)$$

CMB patterns mainly come from here

Evolution for all wavenumbers



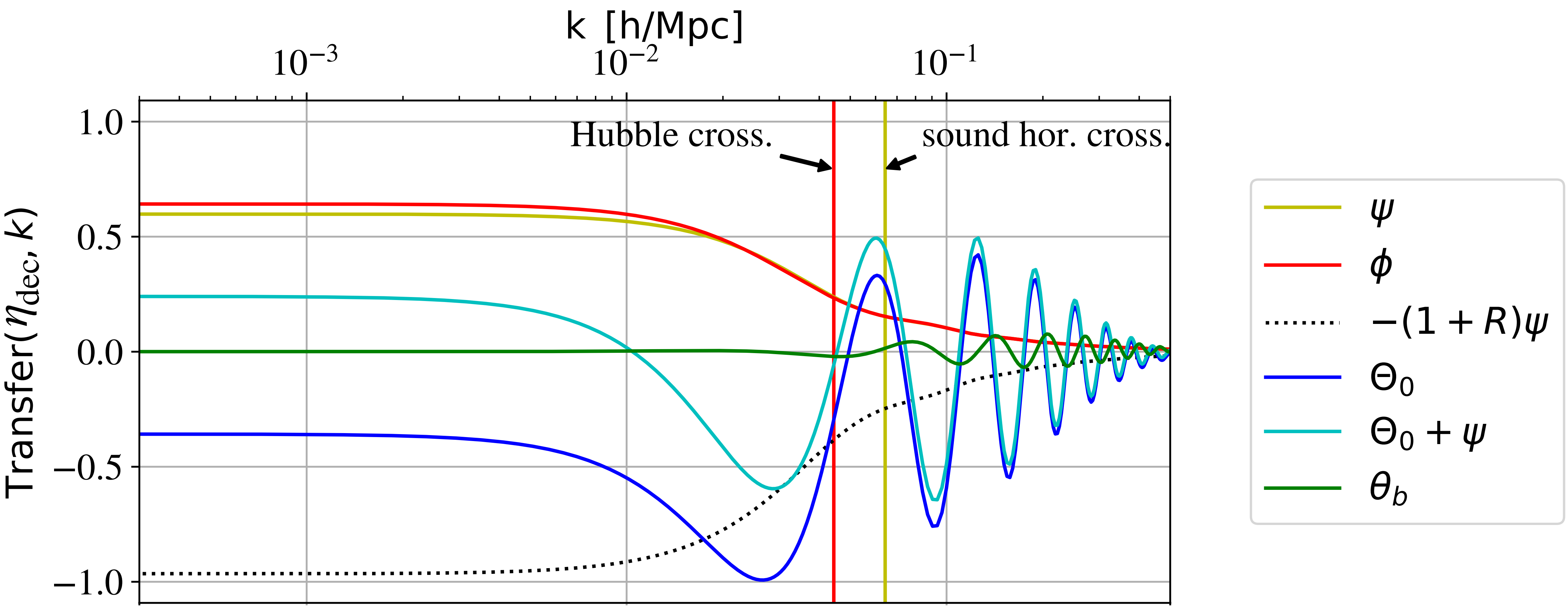
45° symmetry comes from

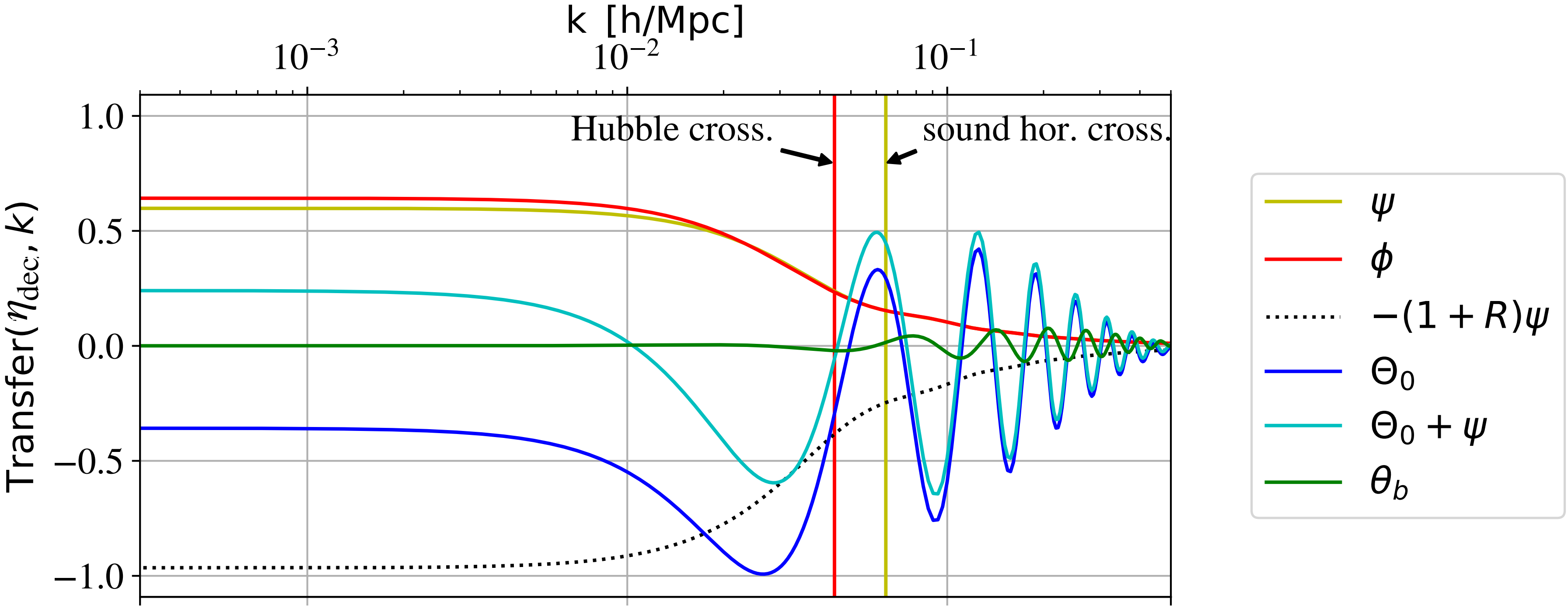
$$\cos(kc_s\eta)$$

CMB patterns mainly come from here

fixed time will look like previous fixed k

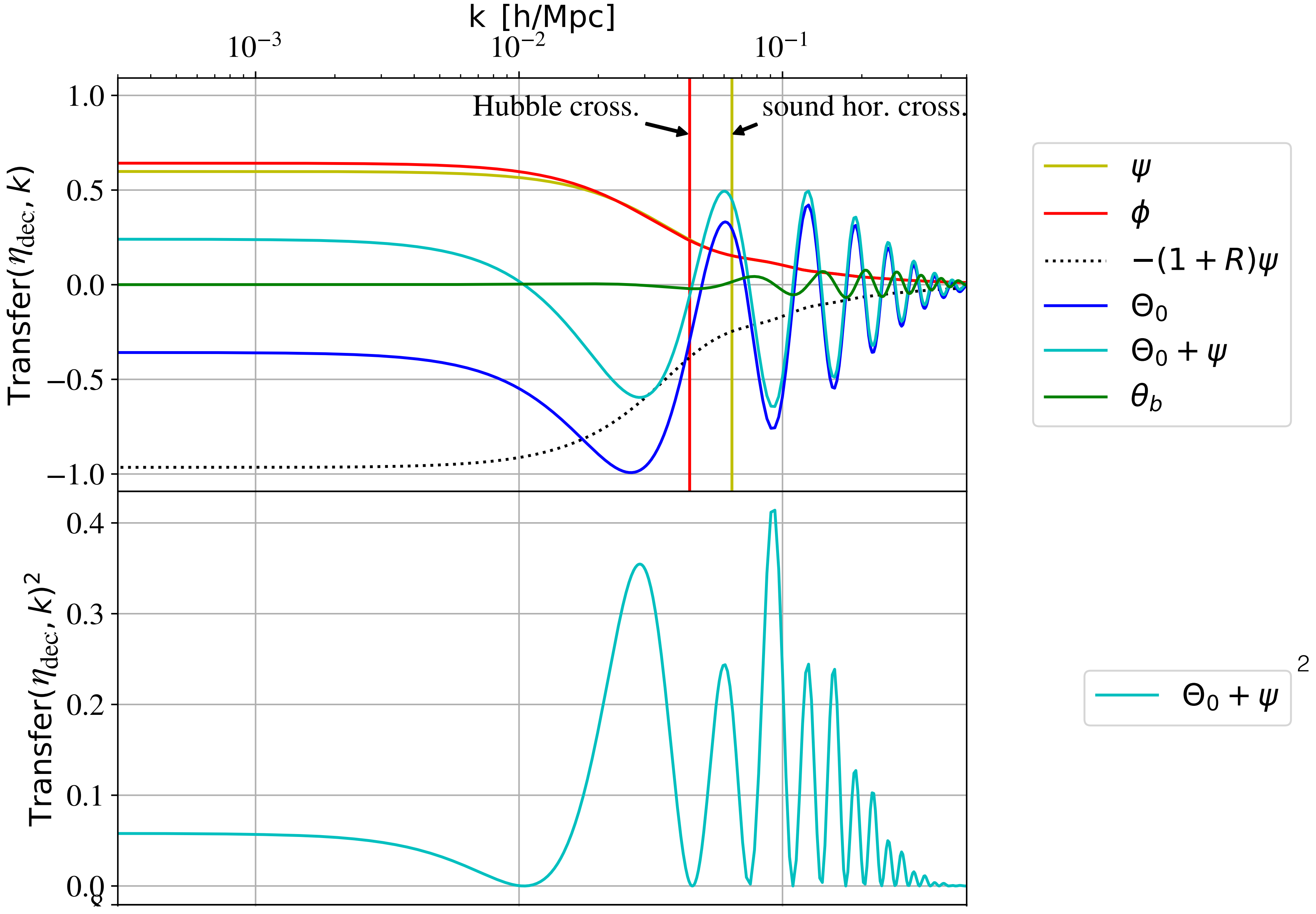
Transfer functions at recombination/decoupling



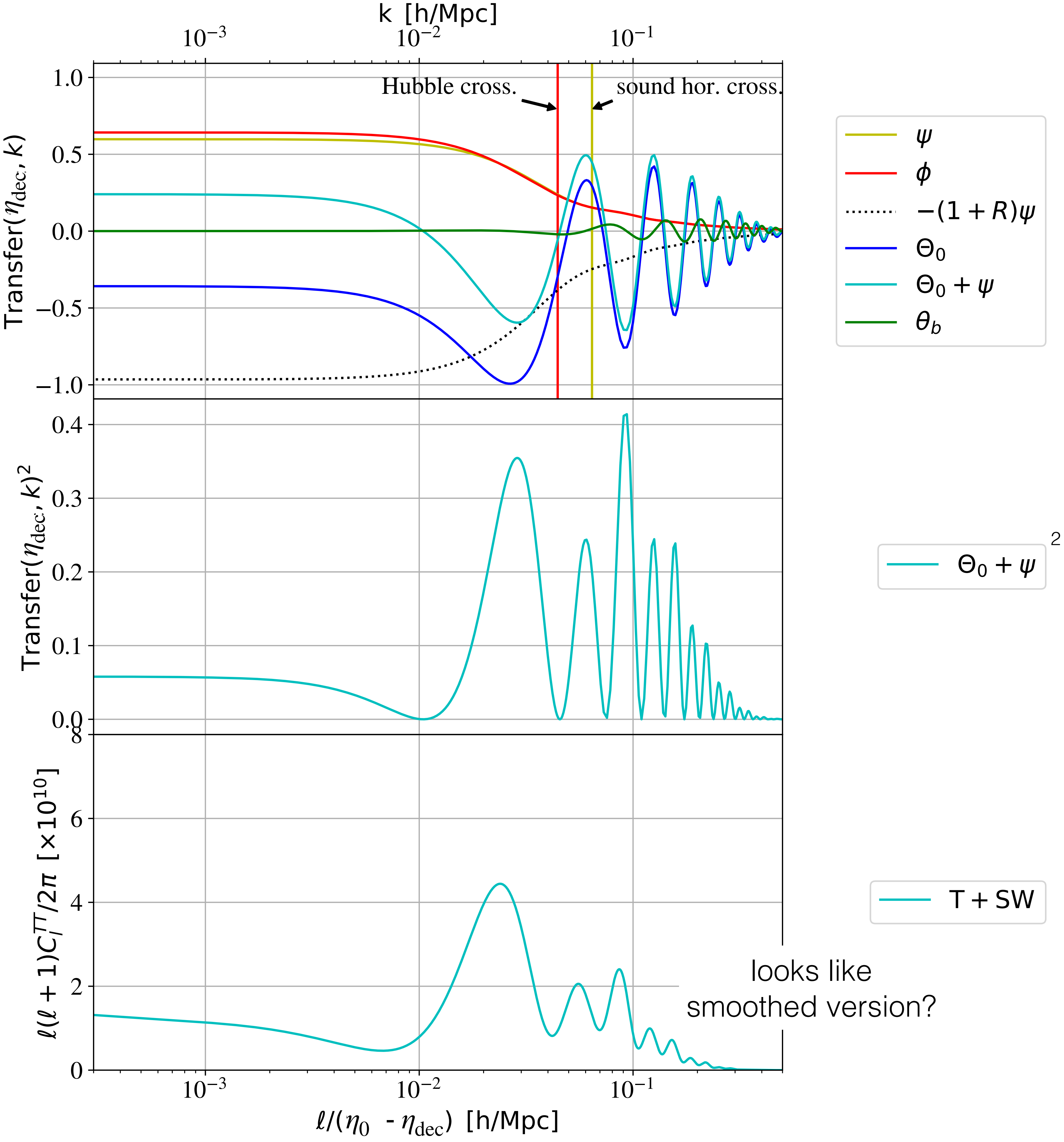


from transfer
to C_ℓ :

from transfer to C_ℓ :



from transfer to C_ℓ :



looks like smoothed version?

$\Theta_0(\eta_{\rm dec}, k) + \psi(\eta_{\rm dec}, k)$

independent of k would give $l(l+1)C_l =$ constant