Recap: Eulerian Perturbation Theory

density $\delta(\underline{x})$, velocity divergence $\theta(\underline{x}) = \underline{\nabla} \cdot \underline{v}(\underline{x})$

linear growth $\delta^{(1)} \sim D(\eta)$, $\theta^{(1)} = -\mathcal{H} f \delta^{(1)}$

higher orders separable, in EdS $\delta^{(2)} \sim D^2(\eta)$, $\theta^{(2)} \sim \mathcal{H} D^2(\eta)$

Lagrangian Perturbation Theory

$$\underline{x}(\underline{q}, \eta) = \underline{q} + \underline{\Psi}(\underline{q}, \eta)$$

$\leftarrow$ displacement field

$$\frac{\partial x_i}{\partial q_j} = \delta_{ij} + \Psi_{i,j} \qquad \det\left(\frac{\partial x_i}{\partial q_j}\right) = J_F$$

$$d^3q = [1 + \delta(\underline{x})]\, d^3x \quad \Rightarrow \quad 1 + \delta(\underline{x}) = J_F^{-1}$$

$$\underline{x}'' + \mathcal{H}\underline{x}' = -\underline{\nabla}_x V$$

$$\underline{\nabla}_x \cdot \left[\partial_\eta^2 \underline{\Psi}(\underline{q}, \eta) + \mathcal{H} \partial_\eta \underline{\Psi}(\underline{q}, \eta)\right] = \frac{3}{2}\mathcal{H}^2 \left[1 - \left(\det \partial x_i / dq_j\right)^{-1}\right]$$

$$\frac{\partial q_j}{\partial x^i} \frac{\partial}{\partial q^j}\left[\partial_\eta^2 \Psi_i(q, \eta) + \mathcal{H} \partial_\eta \Psi_i(q, \eta)\right] = \frac{3}{2}\mathcal{H}^2 \left(1 - J_F^{-1}\right)$$

linearise

$$\partial_\eta^2 \Psi_{i,i}(q, \eta) + \mathcal{H} \partial_\eta \Psi_{i,i}(q, \eta) = \frac{3}{2}\mathcal{H}^2 \left(1 - J_F^{-1}\big|_{\text{lin}}\right)$$

$$J_F = \det\left(\delta_{ij} + \Psi_{i,j}\right) \approx 1 + \underbrace{\Psi_{i,i}}_{\underline{\nabla} \cdot \underline{\Psi}}$$

$$J_F^{-1} \approx \frac{1}{1 + \Psi_{i,i}} \approx 1 - \Psi_{i,i}$$

$$\partial_\eta^2 \Psi_{i,i}(q, \eta) + \mathcal{H} \partial_\eta \Psi_{i,i}(q, \eta) = \frac{3}{2}\mathcal{H}^2 \Psi_{i,i}(q, \eta)$$

separate $\Psi_{i,i}(q, \eta) = D(\eta)\, \tilde{\varphi}_1(\underline{q})$

$\hookrightarrow \tilde{\varphi}_1(q) = -\tilde{\delta}_1(q) = -\nabla^2 \varphi_{\text{ini}}(q)$

nonlinear density from linear LPT

$$1+\delta(\underline{x},\eta) = J_F^{-1} = \frac{1}{\det \partial x_i/\partial q_j}$$

$\uparrow$ $\delta_{ij}+\psi_{i,j}$ $\swarrow D(\eta)\tilde{\psi}_{i,j}(q)$

$$= \frac{1}{(1-D(\eta)\lambda_1)(1-D(\eta)\lambda_2)(1-D(\eta)\lambda_3)}$$

λ_i Eigenvalues $\tilde{\psi}_{i,j}$

Lagrangian tidal tensor

$\lambda_1 \geq \lambda_2 \geq \lambda_3$

different types of collapse

- planar collapse $\lambda_1 > 0$ $\quad \lambda_1 > \lambda_2, \lambda_3$
- filamentary collapse $\lambda_1 > 0$, $\lambda_1 = \lambda_2 > \lambda_3$
- spherical collapse $\lambda_1 > 0$ $\quad \lambda_1 = \lambda_2 = \lambda_3$

$$1+\delta(\underline{x},\eta) = \frac{1}{(1-D(\eta)\lambda_1)^3} = \frac{1}{1-D(\eta)\tilde{\delta}_1(q)}$$

2nd order Lagrangian Perturbation Theory

$$\frac{\partial q_j}{\partial x^i}\frac{\partial}{\partial q^j}\left[\partial_\eta^2 \psi_i(q,\eta) + \mathcal{H}\partial_\eta \psi_i(q,\eta)\right] = \frac{3}{2}\mathcal{H}^2\left(1-J_F^{-1}\right)$$

$$\frac{\partial x^i}{\partial q^j} = \delta_{ij}+\psi_{i,j} \approx \delta_{ij} + \psi_{i,j}^{(1)} + \psi_{i,j}^{(2)}$$

$\frac{\partial q_j}{\partial x^i}$ on the left multiplies ψ

only need linear order

$$\frac{\partial q_j}{\partial x^i} \approx \delta_{ij} - \psi_{i,j}^{(1)}$$

J_F^{-1} on the right we need up to 2nd order

start with J_F

$$J_F = \det\left(\delta_{ij} + \varphi_{i,j}^{(1)} + \varphi_{i,j}^{(2)}\right)$$

$$= 1 + \varphi_{i,i}^{(1)} + \varphi_{i,i}^{(2)} + \frac{1}{2}\left[\varphi_{i,i}^{(1)}\varphi_{j,j}^{(1)} - \varphi_{i,j}^{(1)}\varphi_{j,i}^{(1)}\right]$$

summation convention

$$J_F^{-1} = \frac{1}{1+(J_F-1)} \approx 1-(J_F-1)+(J_F-1)^2$$

$$= 1-\left(\varphi_{i,i}^{(1)} + \varphi_{i,i}^{(2)} + \frac{1}{2}\left[\varphi_{i,i}^{(1)}\varphi_{j,j}^{(1)} - \varphi_{i,j}^{(1)}\varphi_{j,i}^{(1)}\right]\right)$$

$$+ \varphi_{i,i}^{(1)}\varphi_{j,j}^{(1)}$$

$$= 1 - \varphi_{i,i}^{(1)} - \varphi_{i,i}^{(2)} - \frac{1}{2}\left[-\varphi_{i,i}^{(1)}\varphi_{j,j}^{(1)} - \varphi_{i,j}^{(1)}\varphi_{j,i}^{(1)}\right]$$

plug in

$$\frac{\partial q_j}{\partial x^i}\left[\partial_\eta^2\varphi_{i,j}(q,\eta) + \mathcal{H}\partial_\eta\varphi_{i,j}(q,\eta)\right] = \frac{3}{2}\mathcal{H}^2\left(1-J_F^{-1}\right)$$

$$\bigstar\quad \partial_\eta^2\varphi_{i,i}^{(2)} + \mathcal{H}\partial_\eta\varphi_{i,i}^{(2)} - \varphi_{i,j}^{(1)}\left[\partial_\eta^2\varphi_{i,j}^{(1)} + \mathcal{H}\partial_\eta\varphi_{i,j}^{(1)}\right]$$

$$= \frac{3}{2}\mathcal{H}^2\left\{\varphi_{i,i}^{(2)} - \frac{1}{2}\left[+\varphi_{i,i}^{(1)}\varphi_{j,j}^{(1)} + \varphi_{i,j}^{(1)}\varphi_{j,i}^{(1)}\right]\right\}$$

$$\left(\partial_\eta^2 + \mathcal{H}\partial_\eta - \frac{3}{2}\mathcal{H}^2\right)\varphi_{i,i}^{(2)}$$

$$= \varphi_{i,j}^{(1)}\left[\underbrace{\partial_\eta^2\varphi_{i,j}^{(1)} + \mathcal{H}\partial_\eta\varphi_{i,j}^{(1)}}_{3/2\,\mathcal{H}^2\varphi_{i,j}^{(1)}}\right]$$

$$-\frac{3}{2}\mathcal{H}^2\cdot\frac{1}{2}\left[\varphi_{i,i}^{(1)}\varphi_{j,j}^{(1)} + \varphi_{i,j}^{(1)}\varphi_{j,i}^{(1)}\right]$$

separate $\Psi_{i,i}^{(2)}(q,\eta) = D_2(\eta)\,\tilde{\Psi}^{(2)}(q)$

time dependence : focus EdS

ansatz $D_2(\eta) = c \cdot D^2(\eta)$

★ $c\,\partial_\eta^2 D^2(\eta) + c\,\mathcal{H}\partial_\eta D^2(\eta) - \frac{3}{2}\mathcal{H}^2 c\,D^2(\eta)$

$= \frac{3}{2}\mathcal{H}^2 D^2$

$$c\Big[\underbrace{2D'(\eta)^2}_{2\mathcal{H}^2D^2} + \underbrace{2DD''(\eta) + 2\mathcal{H}DD'(\eta)}_{3\mathcal{H}^2D^2} - \frac{3}{2}\mathcal{H}^2D^2\Big] = \frac{3}{2}\mathcal{H}^2D^2$$

$c \cdot \left[2 + 3 - \frac{3}{2}\right] = \frac{3}{2}$

$c \cdot \frac{7}{2} = \frac{3}{2} \quad \Rightarrow \quad c = \frac{3}{7}$

★ $\tilde{\Psi}_{i,i}^{(2)} = -\frac{1}{2}\left(\tilde{\Psi}_{i,i}^{(1)}\tilde{\Psi}_{j,j}^{(1)} - \tilde{\Psi}_{i,j}^{(1)}\tilde{\Psi}_{j,i}^{(1)}\right)$

tidal term $\quad \tilde{\Psi}^{(1)}(q) = -\varphi_{ini}(q)$

we had defined $\Psi^{(1)}_{i,i}(q) = D(\eta)\tilde{\Psi}^{(1)}(q)$ undo

$= -\frac{1}{2}\left(\varphi^{ini}_{,ii}\,\varphi^{ini}_{,jj} - \varphi^{ini}_{,ij}\,\varphi^{ini}_{,ji}\right)$

in 1D this vanishes - no tidal terms

higher order Lagrangian Perturbation Theory

$$\underline{\Psi}^{(n)}(\underline{k}) = i\int \frac{d^3\underline{k_1}\ldots d^3\underline{k_n}}{(2\pi)^{3(n-1)}}\,\delta_D(\underline{k}-\underline{k_1}-\ldots-\underline{k_n})\,\underline{L}^{(n)}(\underline{k_1},\ldots,\underline{k_n})\,\delta_L(\underline{k_1})\ldots\delta_L(\underline{k_n})$$

↑

vector, at low orders it is irrotational extra constraint equation for Eulerian velocity $\underline{v}(\underline{x})$

$$\underline{\nabla}_x \times \underline{v}(x) = 0$$

$$\Leftrightarrow \varepsilon_{ijk} \nabla_{x,j} v_k = 0$$

$$\underline{x}(q,\eta) = \underline{q} + \underline{\Psi}(q,\eta)$$

$$\underline{v}(\underline{x}(q,\eta)) = \partial_\eta \underline{\Psi}(q,\eta)$$

extra constraint $\varepsilon_{ijk} \frac{\partial q_\ell}{\partial x_j} \frac{\partial}{\partial q_\ell} \partial_\eta \Psi_k = 0$

Link to densities

$$\delta(\underline{k}) = \int d^3x \, \exp(-i\underline{k}\cdot\underline{x}) \, \delta(\underline{x})$$

mass conservation $[1+\delta(x)]\, d^3x = d^3q$

$$\delta_D(\underline{x} - \underline{q} - \underline{\Psi})$$

$$\delta(\underline{k}) = \int d^3q \, \exp(-i\underline{k}\cdot\underline{q}) \left[\exp(-i\underline{k}\cdot\underline{\Psi}(\underline{q})) - 1\right]$$