Numerical Cosmology

Lecture 3

Hydrodynamics and Galaxy Formation

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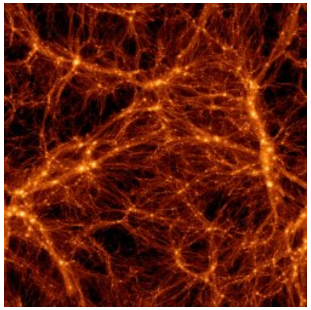
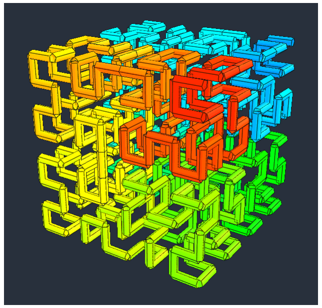
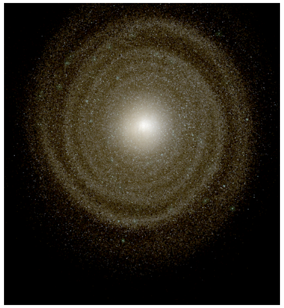






Boltzmann Equation

Baryons are collisional. They are described by Boltzmann equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{g} \cdot \frac{\partial f}{\partial \mathbf{v}} = \int_{4\pi} \int_{\mathbb{R}^3} \left(f_1' f_2' - f_1 f_2 \right) \sigma \left| \mathbf{v}_1 - \mathbf{v}_2 \right| \, \mathrm{d}\Omega \, \mathrm{d}^3 v_2.$$

Collision time: $C_{\rm coll} = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{4\pi} f_1 f_2 \, \sigma v \, \mathrm{d}^3 v_1 \, \mathrm{d}^3 v_2 \, \mathrm{d}\Omega.$

Maxwell-Boltzmann distribution: $f_0(\mathbf{v}) = \frac{\rho}{m} \frac{1}{\left(2\pi\sigma^2\right)^{3/2}} \exp\left(-\frac{1}{2} \frac{\left(\mathbf{v} - \mathbf{u}\right)^2}{\sigma^2} \right).$

Velocity dispersion: $\sigma^2 = \frac{k_B T}{m}.$

Hard sphere model: $\sigma_0 = \pi r_0^2 \simeq 10^{-15} \mathrm{cm}^2.$

$$\frac{1}{\tau_{\rm coll}} = \frac{C_{\rm coll}}{n} \simeq n\sigma_0 \sqrt{\frac{k_B T}{m}} \quad \text{and} \quad \lambda_{\rm coll} = \sqrt{\frac{k_B T}{m}} \tau_{\rm coll} \simeq \frac{1}{n\sigma_0}.$$

Moments of Boltzmann Equation

We define zero-order, first order and second-order moments as:

$$\rho(\mathbf{x},t) = \int_{\mathbb{R}^3} mf \mathrm{d}v^3 \quad \rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t) = \int_{\mathbb{R}^3} m\mathbf{v}f \mathrm{d}v^3 \quad E(\mathbf{x},t) = \int_{\mathbb{R}^3} \frac{1}{2}mv^2 f \mathrm{d}v^3.$$

Non-LTE equations:

$$\frac{\partial}{\partial t}\rho + \nabla\cdot(\rho\mathbf{u}) = 0$$

$$\frac{\partial}{\partial t}(\rho\,\mathbf{u}) + \nabla\cdot(\rho\,\mathbf{u}\otimes\mathbf{u} + \mathbb{P}) = \rho\,\mathbf{g}$$

$$\frac{\partial E}{\partial t} + \nabla\cdot(E\,\mathbf{u} + \mathbb{P}\mathbf{u} + \mathbf{Q}) = \rho\,\mathbf{g}\cdot\mathbf{u}$$

Total energy:

$$E = \frac{1}{2}\rho u^2 + e$$

Internal energy: e

Thermal particle velocity: $\mathbf{w} = \mathbf{v} - \mathbf{u}(\mathbf{x},t)$

Pressure tensor: $P_{ij} = \int_{\mathbb{R}^3} m w_i w_j f \mathrm{d}^3 v$

Heat flux: $\mathbf{Q} = \int_{\mathbb{R}^3} m\frac{1}{2}w^2\mathbf{w}f\,\mathrm{d}^3 v$

The Euler-Poisson Equations

$$\frac{\partial}{\partial t}\rho + \nabla\cdot(\rho\mathbf{u}) = 0$$

$$\frac{\partial}{\partial t}(\rho\,\mathbf{u}) + \nabla\cdot(\rho\,\mathbf{u}\otimes\mathbf{u} + P\mathbb{I}) = \rho\,\mathbf{g}$$

$$\frac{\partial E}{\partial t} + \nabla\cdot(E+P)\mathbf{u} = \rho\mathbf{g}\cdot\mathbf{u} \text{ with } P = \frac{\rho k_B T}{m}$$

Under strict LTE conditions:

$$H_P = P\left(\frac{\partial P}{\partial x}\right)^{-1} \gg \lambda_{\rm coll}$$

Chapman-Enskog theory (not too far from LTE):

Pressure tensor: $\mathbb{P} = P\mathbb{I} - \mu\left(\mathbb{G} + \mathbb{G}^{\rm T} - \frac{2}{3}(\nabla\cdot\mathbf{u})\mathbb{I}\right)$ with $G_{ij} = \frac{\partial u_i}{\partial x_j}$

Heat flux: $\mathbf{Q} = -\kappa\nabla T$

Viscosity coefficient: $\mu = \rho\lambda_{\rm coll}\sqrt{\frac{k_B T}{m}} = \rho\nu_{\rm coll}$

Conduction coefficient: $\kappa = \rho\lambda_{\rm coll}\sqrt{\frac{k_B T}{m}}\frac{k_B}{m} = \rho\frac{k_B}{m}\nu_{\rm coll}$

Microscopic diffusion:

$$\nu_{\rm coll} = \lambda_{\rm coll}\sqrt{\frac{k_B T}{m}}$$

Finite Volume Scheme

We discretize 1D space with finite volumes $V_i = [x_{i-1/2}, x_{i+1/2}]$.

We define the vector of conservative variables $\mathbf{U} = (\rho, \rho u, E)$.

The Euler equations have a conservative form: $\dfrac{\partial \mathbf{U}}{\partial t} + \dfrac{\partial \mathbf{F}}{\partial x} = 0$,

with the vector of flux functions $\mathbf{F} = \left(\rho u, \rho u^2 + P, u(E+P)\right)$.

Integrating both in space between $x_{i-1/2}$ and $x_{i+1/2} = x_{i-1/2} + \Delta x$ and

in time between t^n and $t^{n+1} = t^n + \Delta t$, we get the discrete integral form:

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i+1/2}^{n+1/2} - \mathbf{F}_{i-1/2}^{n+1/2}\right)$$

where $\mathbf{U}_i^n = \dfrac{1}{\Delta x}\displaystyle\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{U}(x, t^n)\mathrm{d}x$ and $\mathbf{F}_{i+1/2}^{n+1/2} = \dfrac{1}{\Delta t}\displaystyle\int_{t^n}^{t_{n+1}} \mathbf{F}(x_{i+1/2}, t)\mathrm{d}t$.

Question: how do we compute the numerical flux $\mathbf{F}_{i+1/2}^{n+1/2}$?

Godunov Method

The flux $\mathbf{F}_{i+1/2}^{n+1/2}$ is defined at the interface between 2 piecewise contant states: $\mathbf{U}_L = \mathbf{U}_i^n$ and $\mathbf{U}_R = \mathbf{U}_{i+1}^n$. This is called a Riemann problem.

Godunov's idea: solve the Riemann problem (even approximately) and evaluate the flux at the interface $x_{i+1/2}$. This flux is constant in time.

$$\mathbf{F}_{i+1/2}^{n+1/2} = \mathrm{RP}(\mathbf{U}_L, \mathbf{U}_R).$$

The key component is the Riemann solver:

- Exact Riemann solver
- Harten-Lax-van Leer (HLL) Riemann solver
- HLLC (Toro) Riemann solver
- Lax-Friedrich Riemann solver:

$$\mathbf{F}_{i+1/2}^{n+1/2} = \frac{1}{2}\left(\mathbf{F}_L + \mathbf{F}_R\right) - \frac{c_{\max}}{2}\left(\mathbf{U}_R - \mathbf{U}_L\right) \text{ where } c_{\max} = \max_{L,R}\left(|u| + c_s\right)$$

Advection Equation

We use as an example the advection equation with velocity $u = a$ constant.

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = 0.$$

Conservative update: $\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}\left(f_{i+1/2}^{n+1/2} - f_{i-1/2}^{n+1/2}\right)$.

We use the upwind flux (solution to the Riemann problem):

$$f_{i+1/2}^{n+1/2} = a\rho_i^n \quad \text{if} \quad a > 0.$$

$$f_{i+1/2}^{n+1/2} = a\rho_{i+1}^n \quad \text{if} \quad a < 0.$$

For $a > 0$, we get: $\rho_i^{n+1} = \rho_i^n(1 - C) + \rho_{i-1}^n C$. The new solution is a convex combination of the old solution: monotonicity, positivity, stability.

Courant-Friedrich-Levy stability condition: $C = a\frac{\Delta t}{\Delta x} < 1.$

Modified Equation

We have the numerical update for $a > 0$: $\rho_i^{n+1} = \rho_i^n(1 - C) + \rho_{i-1}^n C$.

We perform a Taylor expansion both in space and time:

$$\rho_{i-1}^n = \rho_i^n - \Delta x \frac{\partial \rho}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 \rho}{\partial x^2} + \mathcal{O}(\Delta x^3)$$

$$\rho_i^{n+1} = \rho_i^n + \Delta t \frac{\partial \rho}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 \rho}{\partial t^2} + \mathcal{O}(\Delta t^3)$$

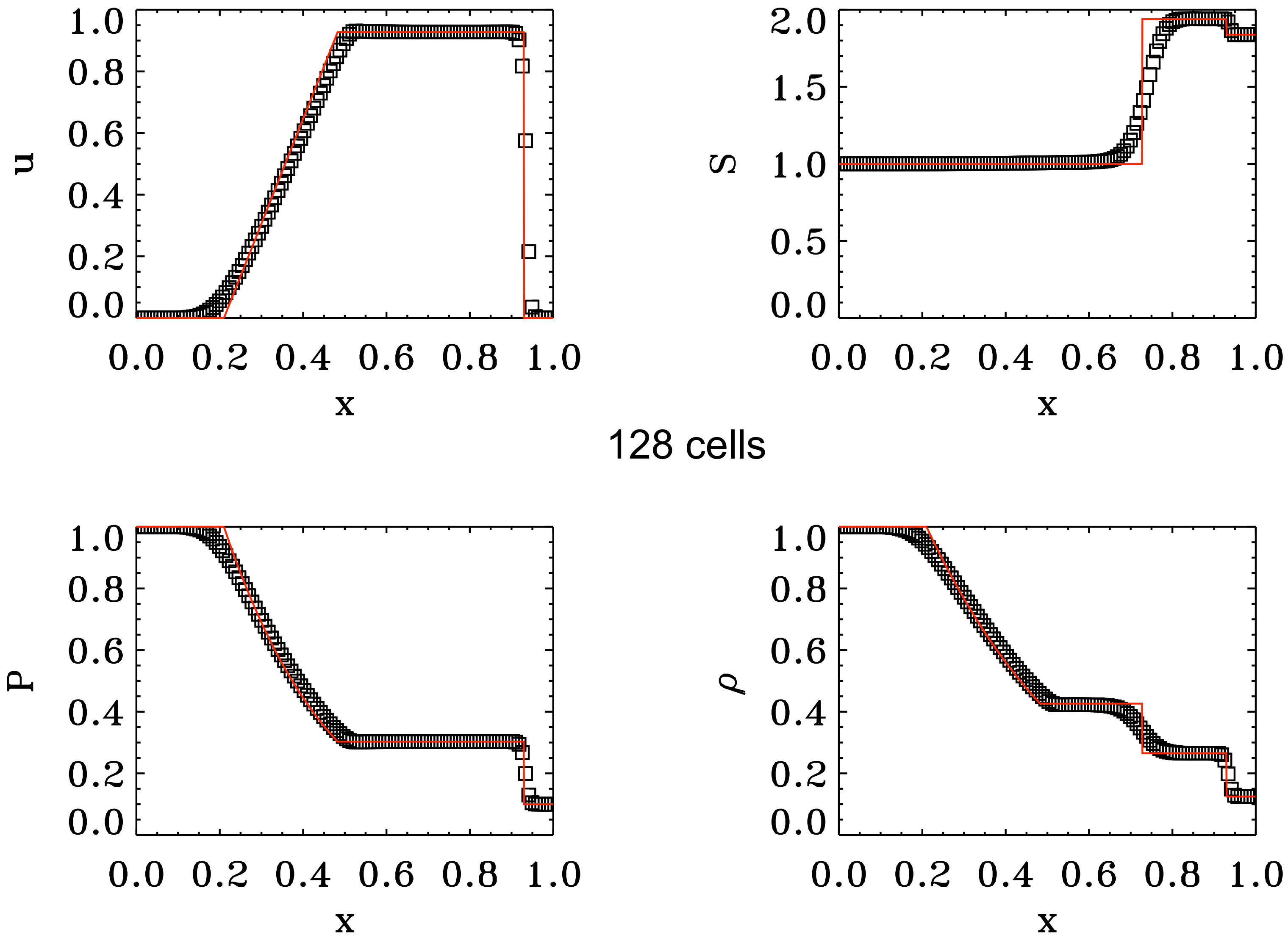
We get $\dfrac{\partial \rho}{\partial t} + \dfrac{\Delta t}{2} \dfrac{\partial^2 \rho}{\partial t^2} = -a \dfrac{\partial \rho}{\partial x} + a \dfrac{\Delta x}{2} \dfrac{\partial^2 \rho}{\partial x^2}$.

We use: $\dfrac{\partial^2 \rho}{\partial t^2} = a^2 \dfrac{\partial^2 \rho}{\partial x^2}$ and get the *modified equation*:

$$\frac{\partial \rho}{\partial t} + a \frac{\partial \rho}{\partial x} = \frac{a \Delta x}{2}(1 - C) \frac{\partial^2 \rho}{\partial x^2} + \mathcal{O}(\Delta x^2, \Delta t^2).$$

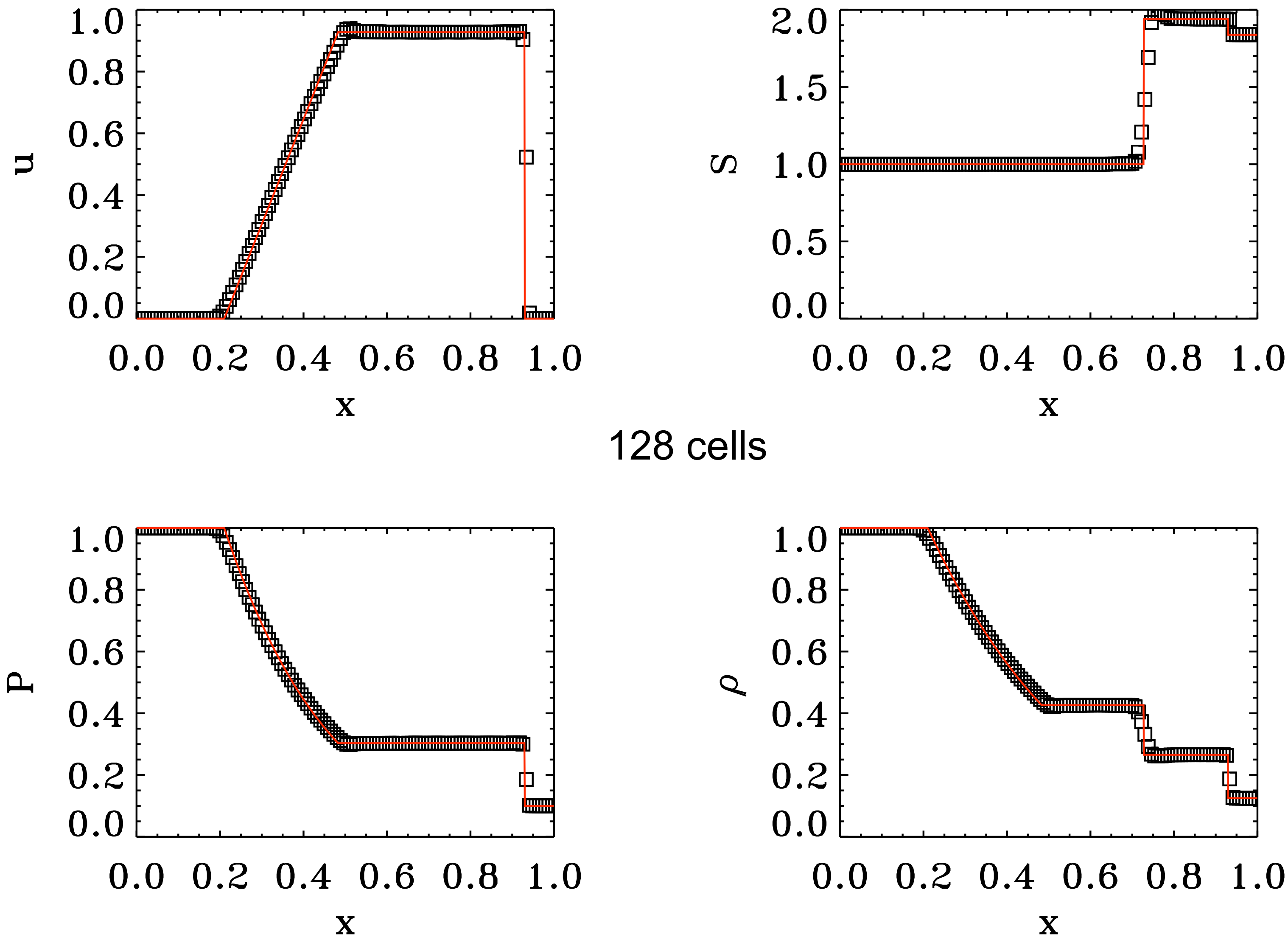
Numerical diffusion: $\nu_{\rm num} \simeq a \Delta x$ compared to $\nu_{\rm coll} = \lambda_{\rm coll} \sqrt{\dfrac{k_B T}{m}}$.

Sod test with first order Godunov scheme

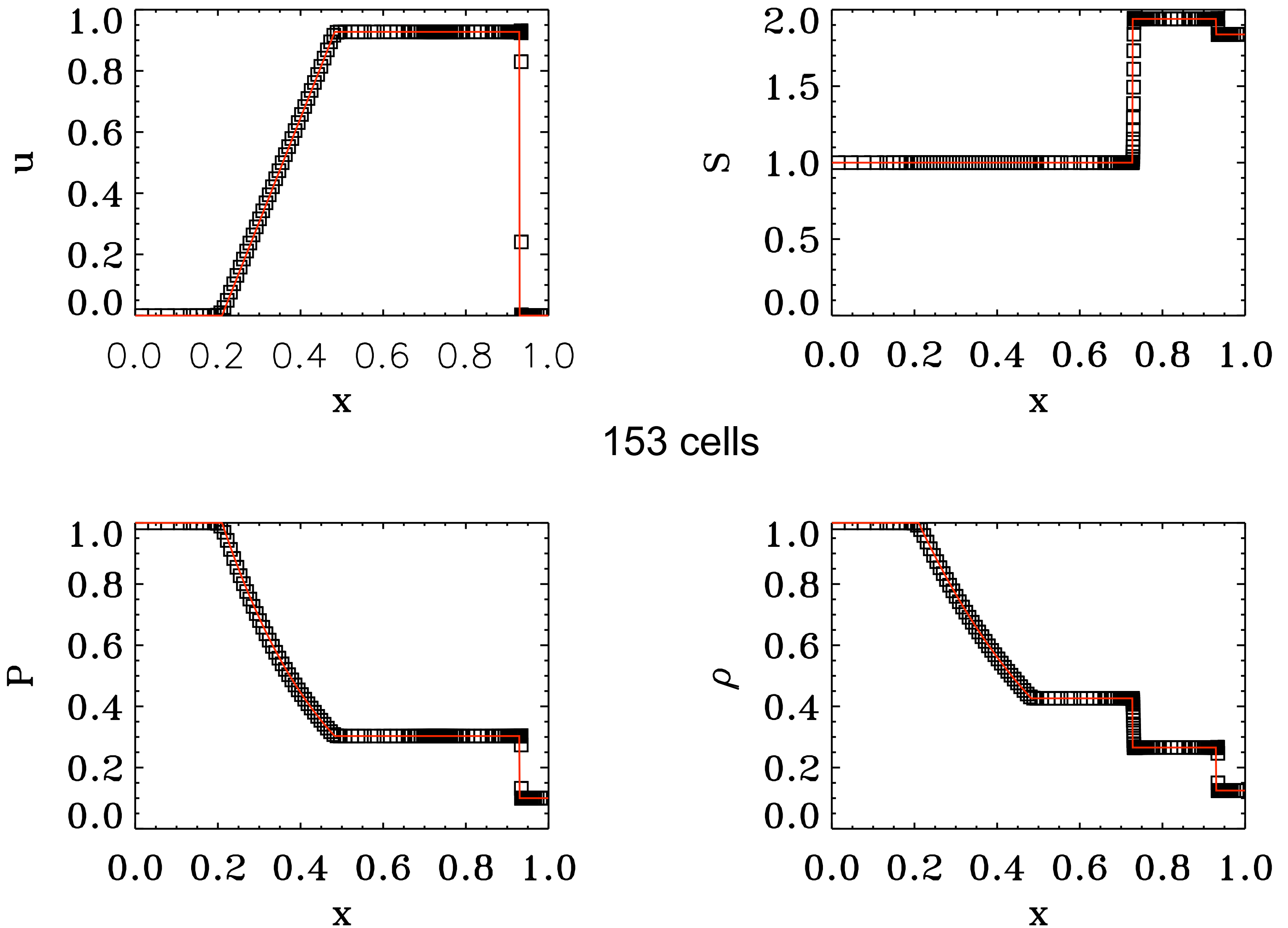


128 cells

Sod test with second order Godunov scheme



Sod test with 2nd order Godunov + AMR



Shock Heating and Hydrostatic Equilibrium

In the spherical collapse model, gas infall will be halted by an accretion shock.

We can write in the frame of the shock the Rankine-Hugoniot relations:

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2$$

$$\left(\frac{1}{2}\rho_1 u_1^2 + \frac{\gamma}{\gamma - 1} P_1\right) u_1 = \left(\frac{1}{2}\rho_2 u_2^2 + \frac{\gamma}{\gamma - 1} P_2\right) u_2$$

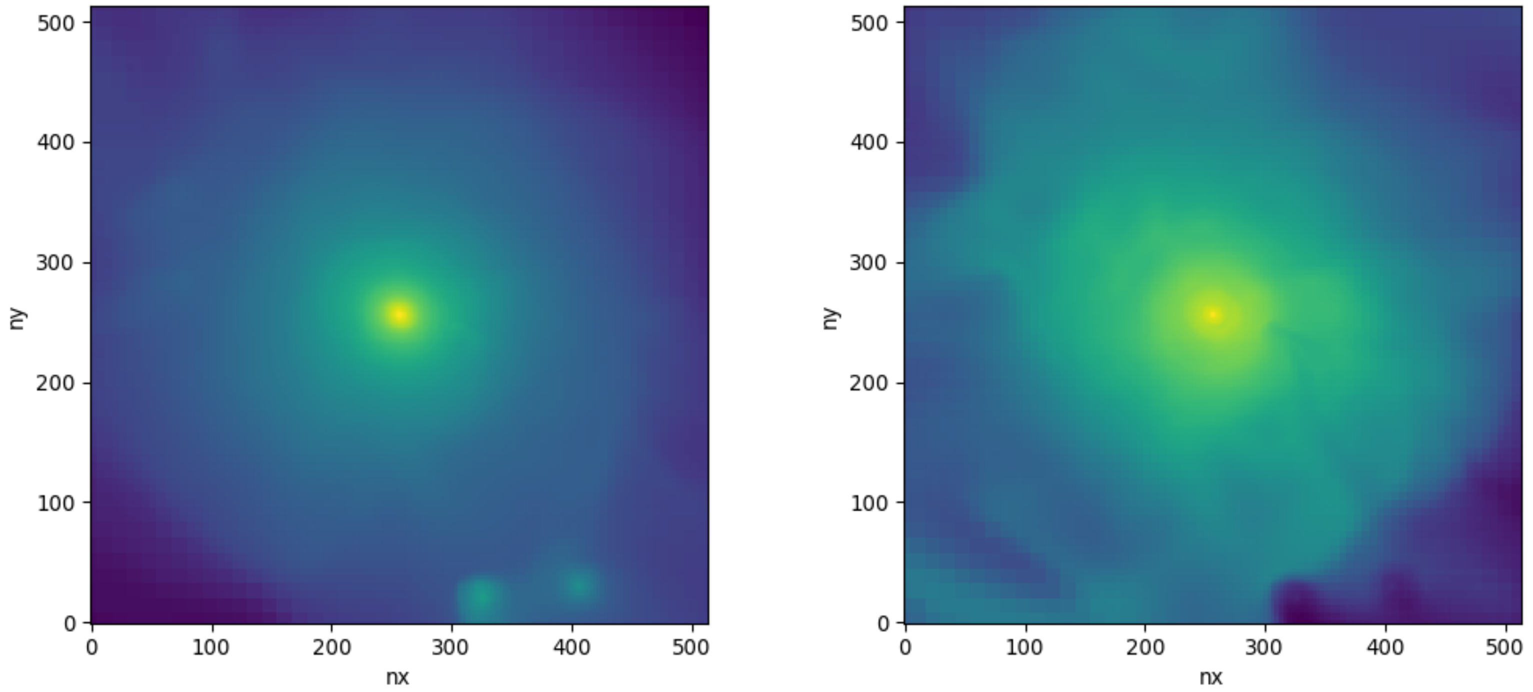
In case of a strong shock $P_2 \ll \rho_2 u_2^2$, we find in the virialized region $P_1 \gg \rho_1 u_1^2$:

$$\frac{k_B T_{\rm vir}}{m} \simeq u_{\rm vir}^2 \quad \text{with} \quad u_{\rm vir}^2 \simeq \frac{GM}{R_{\rm vir}} \simeq V_{\rm circ}^2.$$

In practice, gas infall is more complex that the spherical collapse model so we have non-uniform density and temperature profiles.

Non-Radiative Cosmological Simulations

Projected gas density (left) and temperature (right) in the halo (500kpc across).



Hydrostatic Equilibrium Model

We solve the hydrostatic equation, assuming NFW and a polytropic relation:

$$\frac{1}{\rho}\frac{\partial P}{\partial r} = -\frac{GM_{\rm tot}(r)}{r^2} \quad \text{and} \quad P(r) = P_0\left(\frac{\rho(r)}{\rho_0}\right)^{\Gamma}.$$

We find the Komatsu & Seljak (2001) solution:

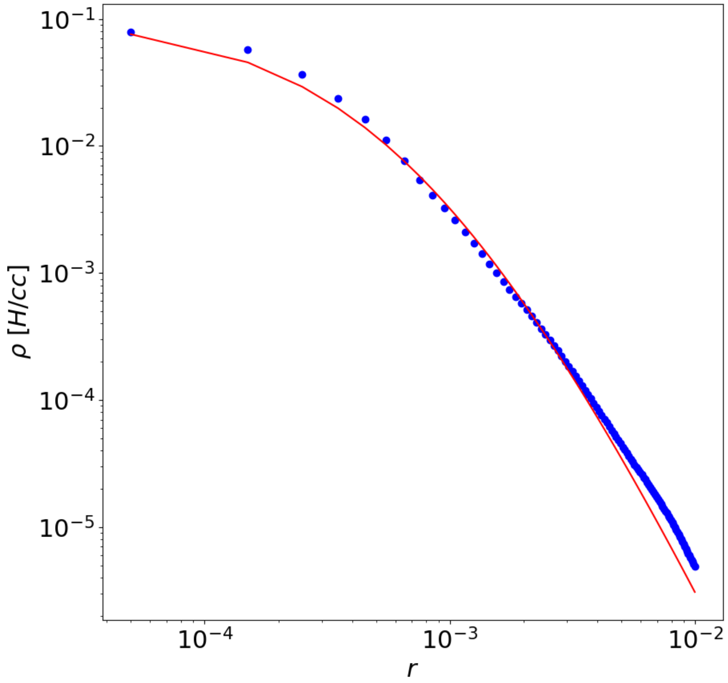
$$T(r) = T_0\frac{\ln 1+x}{x} \quad \text{and} \quad \rho(r) = \rho_0\left(\frac{\ln 1+x}{x}\right)^{\frac{1}{\Gamma-1}} \quad \text{with} \quad x = \frac{r}{r_s}.$$

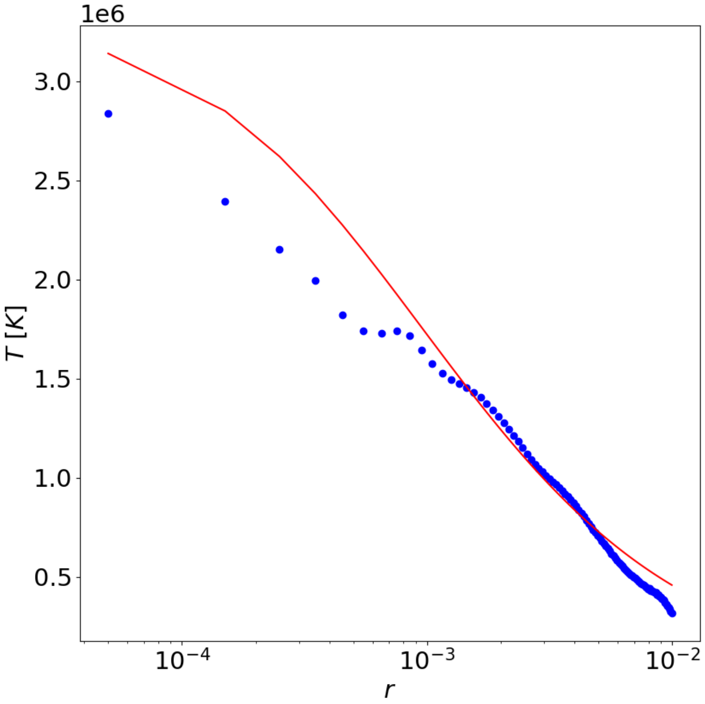
We have $\dfrac{k_B T_0}{m} = 4\pi G\rho_s r_s^2\dfrac{\Gamma-1}{\Gamma}$ and $\rho_0 \simeq 0.2\rho_s$.

The profile are adjusted with only one free parameter $\Gamma \simeq 1.19$ in most non-radiative cosmological simulation across a wide range of masses.

Hydrostatic Equilibrium Model

Spherical density (left) and temperature (right) profiles of the halo.





Radiative Cooling

In the previous collision integral, we only considered elastic collisions.

Some of these collisions are in fact inelastic.

Electrons and protons collide and emit a recombination photon with a small probability $P_{\rm rec}$.

$\rm e^- + H^+ \rightarrow H^0 + \gamma$ with photon energy $h\nu_{\rm rec} \simeq 13.6$ eV.

Assuming a simple hard sphere model, we compute the emission rate:

$Q_{\rm rec} \simeq n_{\rm e^-} n_{\rm H^+} \sigma_0 \sqrt{\frac{k_B T}{m}} P_{\rm rec} h\nu_{\rm rec} = n_H^2 \Lambda(T)$ where $\Lambda(T)$ is the cooling function.

We find: $\Lambda(T) \simeq 10^{-22} \sqrt{\frac{T}{10^4 \ K}}$ erg s^{-1}cm^3 but it drops to zero for $T < 10^4$ K.

We can now estimate the cooling time of a halo of mass $M_{\rm vir}$:

$$t_{\rm cool} = \frac{\frac{3}{2} n_H k_B T_{\rm vir}}{Q_{\rm rec}(T_{\rm vir})} \simeq 100 \left(\frac{T_{\rm vir}}{10^4 \ K} \right)^{1/2} \left(\frac{n_H}{10^{-5} \ {\rm cm}^{-3}} \right)^{-1} {\rm Myr}.$$

Cooling function for astrophysical plasmas

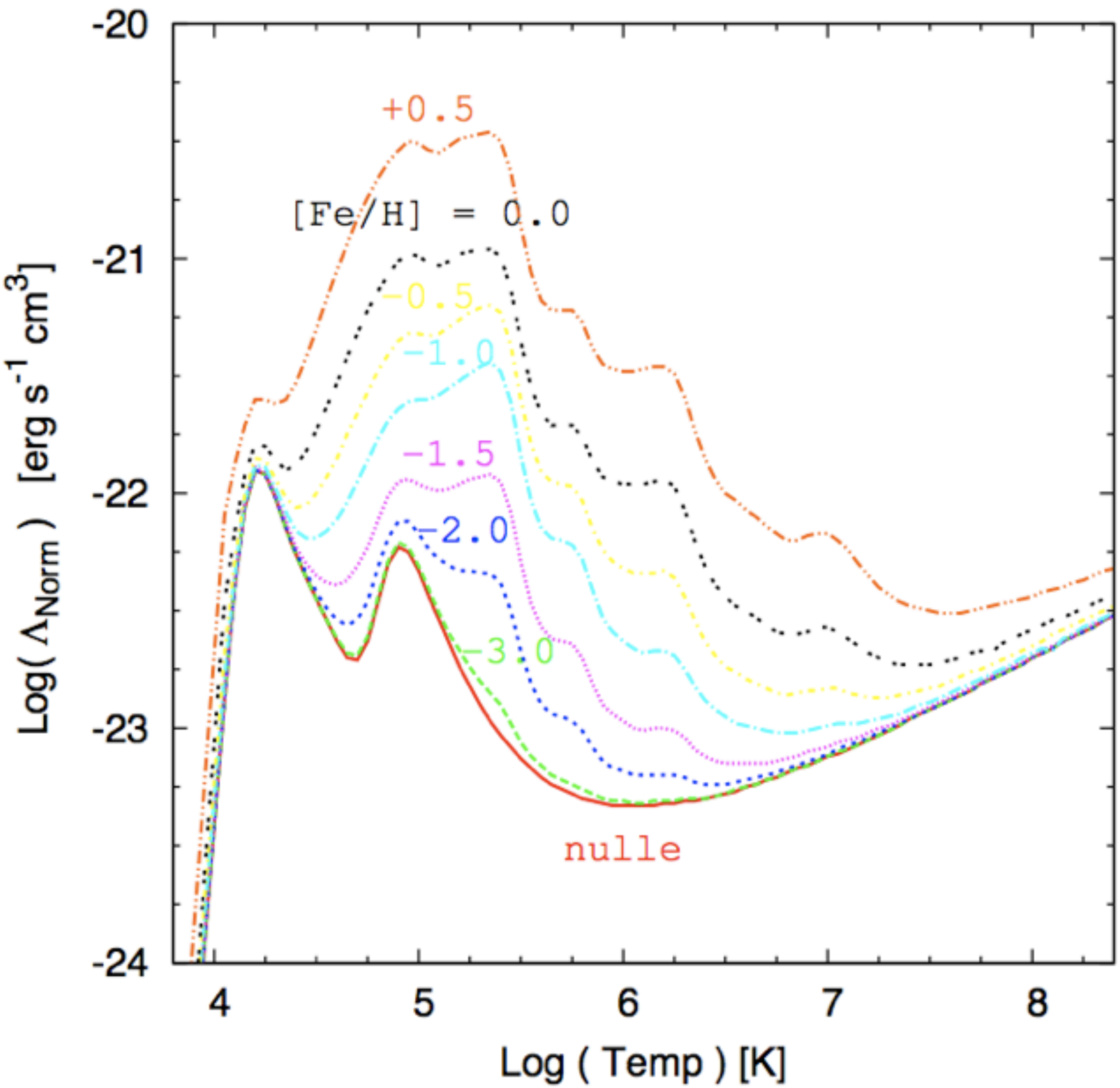
Radiation is emitted or absorbed when electrons make transitions between different states:

Bound-bound: electrons moves between 2 bound states in an atom or an ion. A photon is emitted or absorbed.

Bound-free: electrons move to the continuum (ionization) or a absorbed from the continuum to a bound state (recombination)

Free-free: electrons in the continuum gain or loose energy (a photon) when orbiting around ions (Bremsstrahlung).

Collisional Ionization Equilibrium: depends only on T



Centrifugal Equilibrium and Disk Formation

For halos with $10^4 < T_{\rm vir} < 10^6$ K and $10^9 < M_{\rm vir} < 10^{12}$ $\mathrm{M}_\odot$, we get:

$$t_{\rm cool} < t_{\rm dyn} = \frac{R_{\rm vir}}{V_{\rm vir}} \simeq 1 \ \mathrm{Gyr}.$$

Hydrostatic equilibrium and pressure support are broken and the gas collapses until a new equilibrium is reached: centrifugal equilibrium.

$$\frac{v_\theta^2}{r} = \frac{GM_{\rm tot}(r)}{r^2} \quad \text{or} \quad v_\theta^2 = \frac{GM_{\rm tot}(r)}{r} \simeq V_{\rm circ}^2.$$

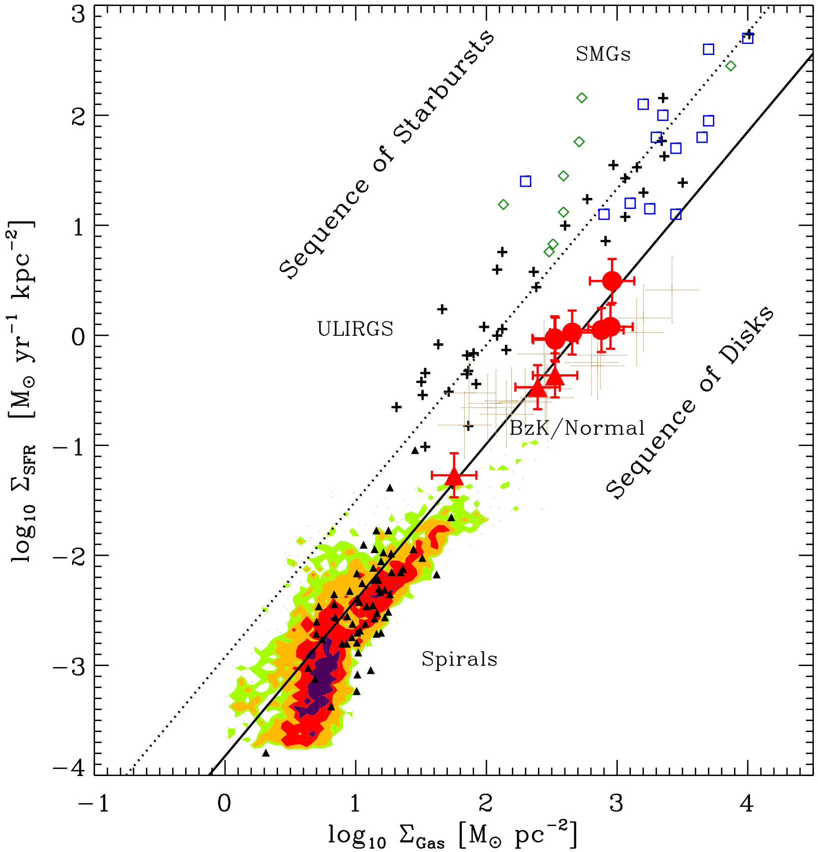
The origin of the final disk size is poorly understood.

Empirical relation: $r_{\rm disk} \simeq 0.015 \ R_{\rm vir}$ (Kravtsov 2013)

Simple Star Formation Recipe

Empirical relation for nearby galaxies: Kennicutt relation.

$$\Sigma_{SFR} = (2.5 \pm 0.7) \times 10^{-4} \left(\frac{\Sigma_{gas}}{M_\odot pc^{-2}} \right)^{1,4} [M_\odot yr^{-1} kpc^{-2}].$$



Star formation recipe: Schmidt law:

$$\dot{\rho}_* = \epsilon_{ff} \frac{\rho}{t_{ff}}$$

Free-fall time: $\frac{1}{t_{ff}} = \sqrt{4\pi G \rho}$.

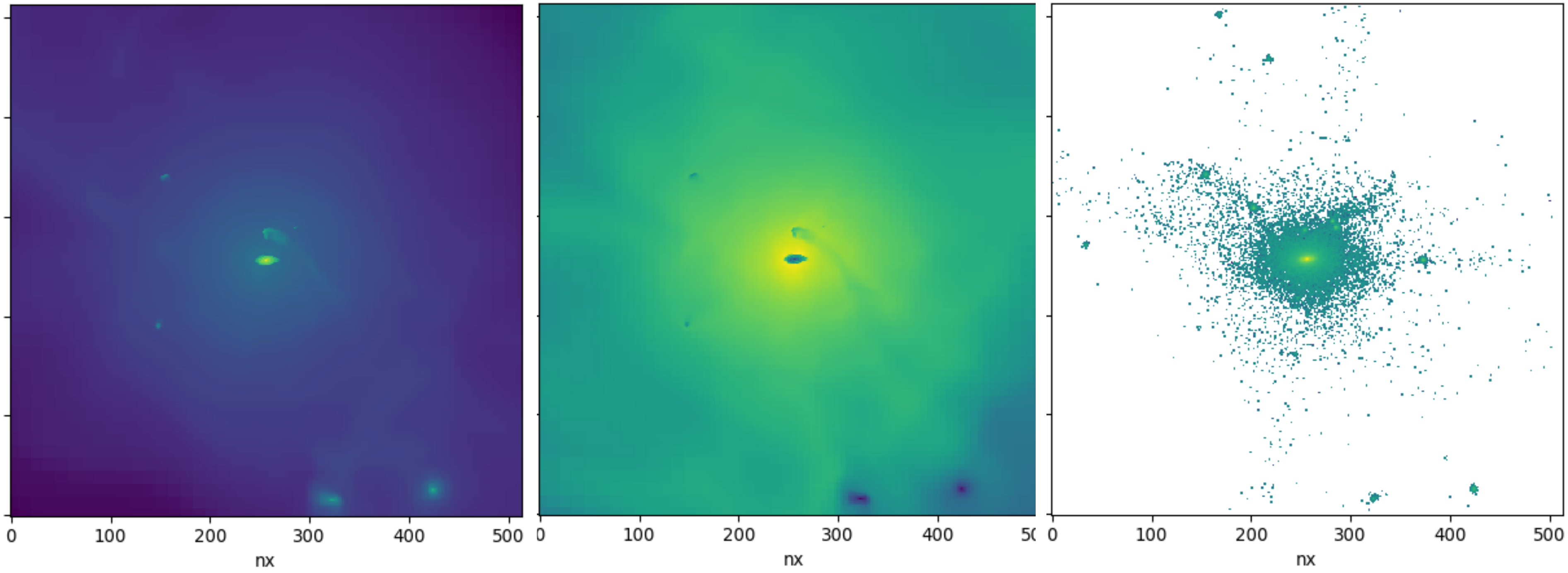
Density threshold: $\rho > \rho_*$

Choose for each simulation the 2 parameters ρ_* and ϵ_{ff} to fit Kennicutt's relation (calibration).

Typical values: $\epsilon_{ff} \simeq 0.01$ and $\rho_* \simeq 0.1 \text{ to } 10 \text{ H/cc}$.

Cooling and Star Formation Simulations

Projected gas density (left), temperature (middle) and star particles (500kpc).



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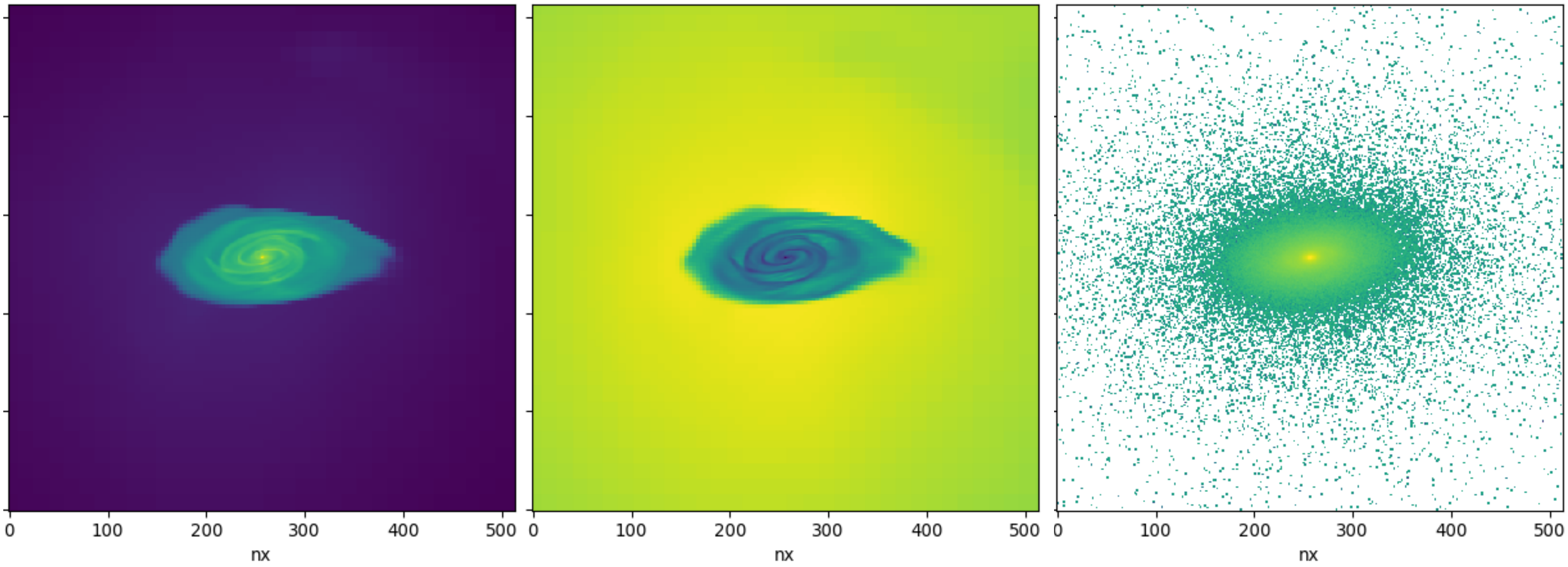


Cooling and Star Formation Simulations

Simulation Michael Kretschmer

The stellar mass problem

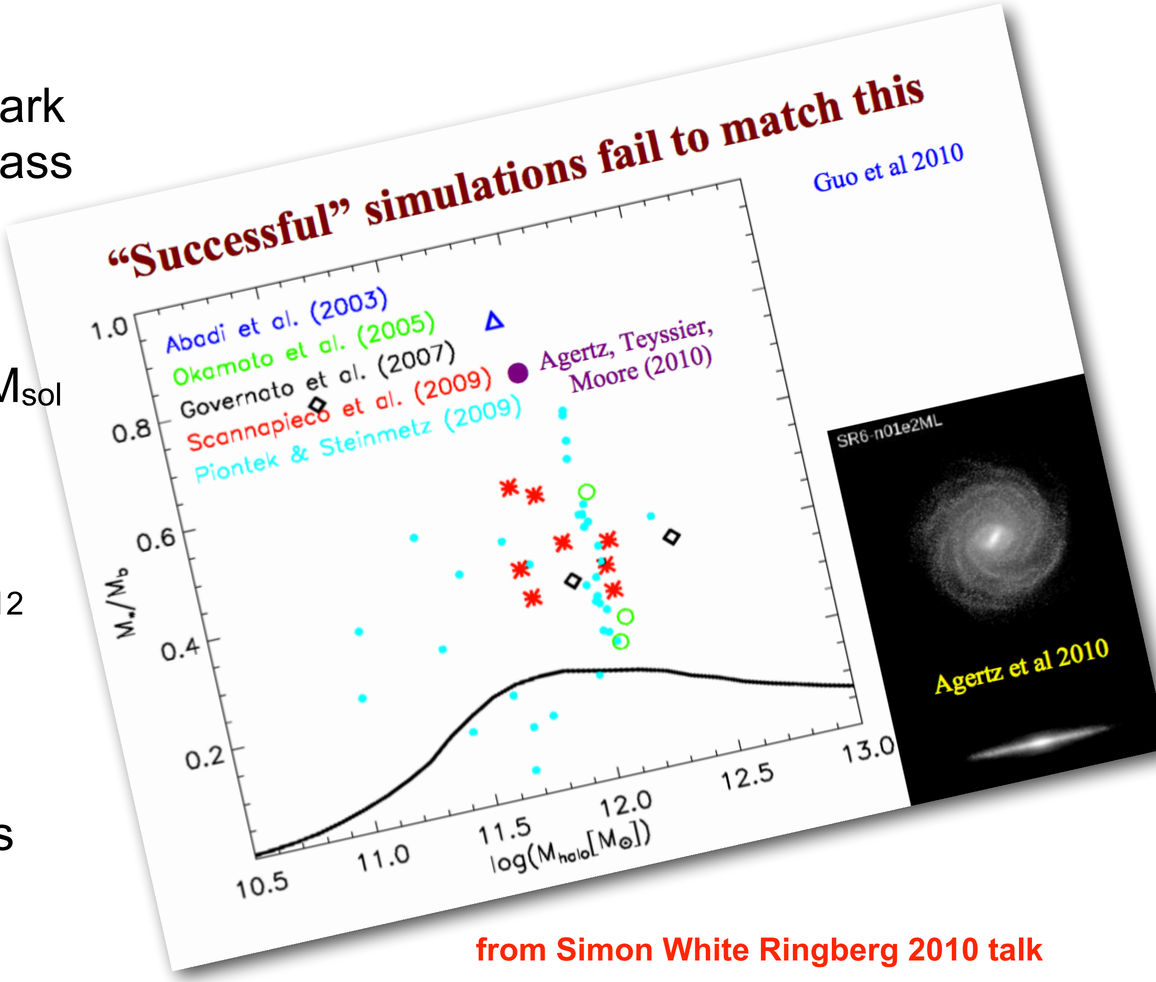
Using abundance matching with dark halos, one can relate the stellar mass to the halo mass.

Berhoozi et al. (2013) $M_{halo}=10^{12}$ M_{sol} for the Milky Way and 25% SFE.

Our simulation suggests $M_{halo}=10^{12}$ M_{sol} but 80% SFE.

Low baryon fraction in MW models requires very efficient feedback.

Key missing ingredient.



from Simon White Ringberg 2010 talk