ÉCOLE DE PHYSIQUE DES HOUCHES

Summer School 2025: The Dark Universe

Bridging perturbation theory and simulations:

Initial conditions and fast integrators for cosmological simulations

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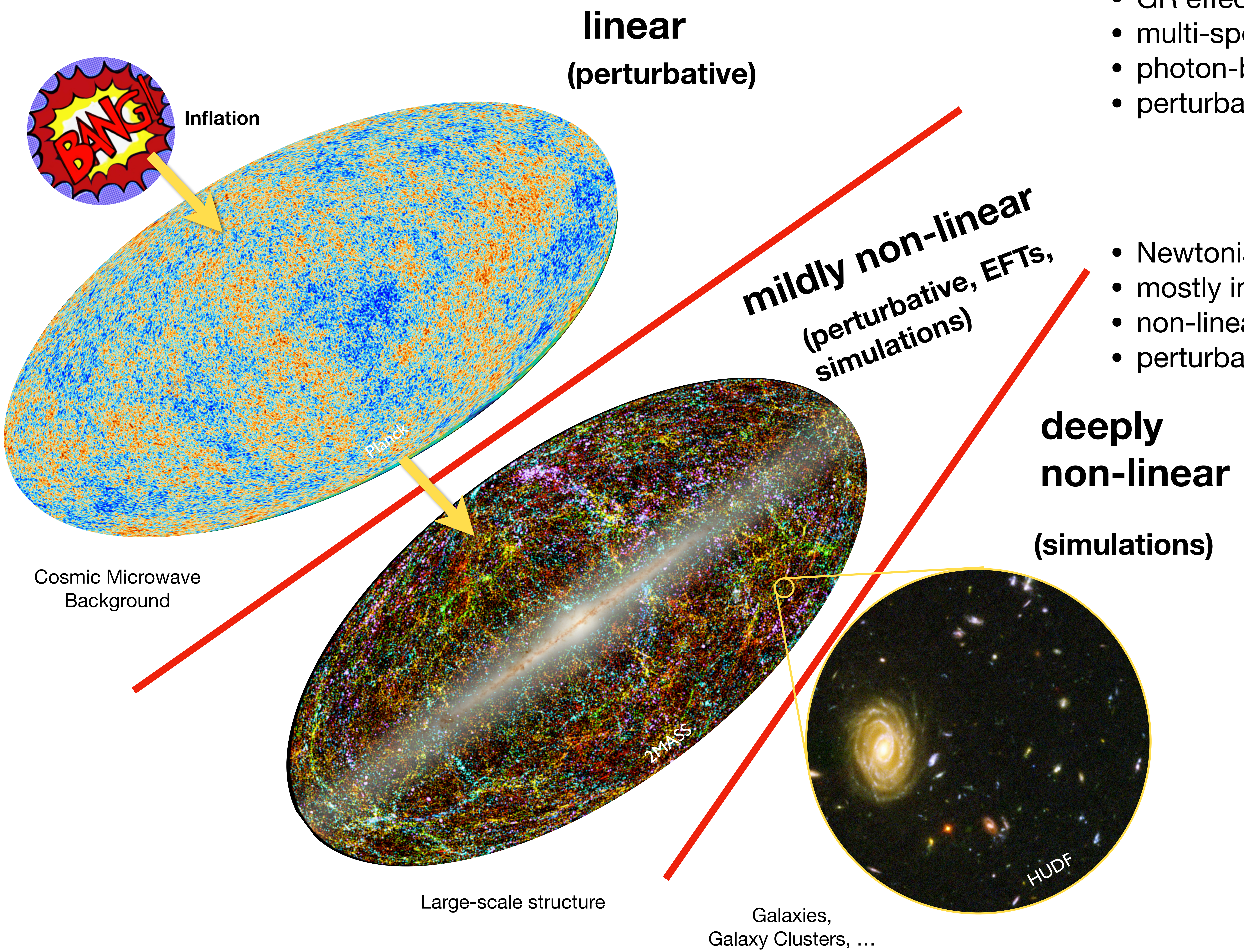
The cosmic history model



- GR effects (horizon+rel. species+aniso-stress)
- multi-species (CDM+baryon+photons+neutrinos)
- photon-baryon coupling + recombination
- perturbative quantity: δ and θ

- Newtonian gravity + small corrections
- mostly interested in mass distribution, CDM+baryons
- non-linear growth
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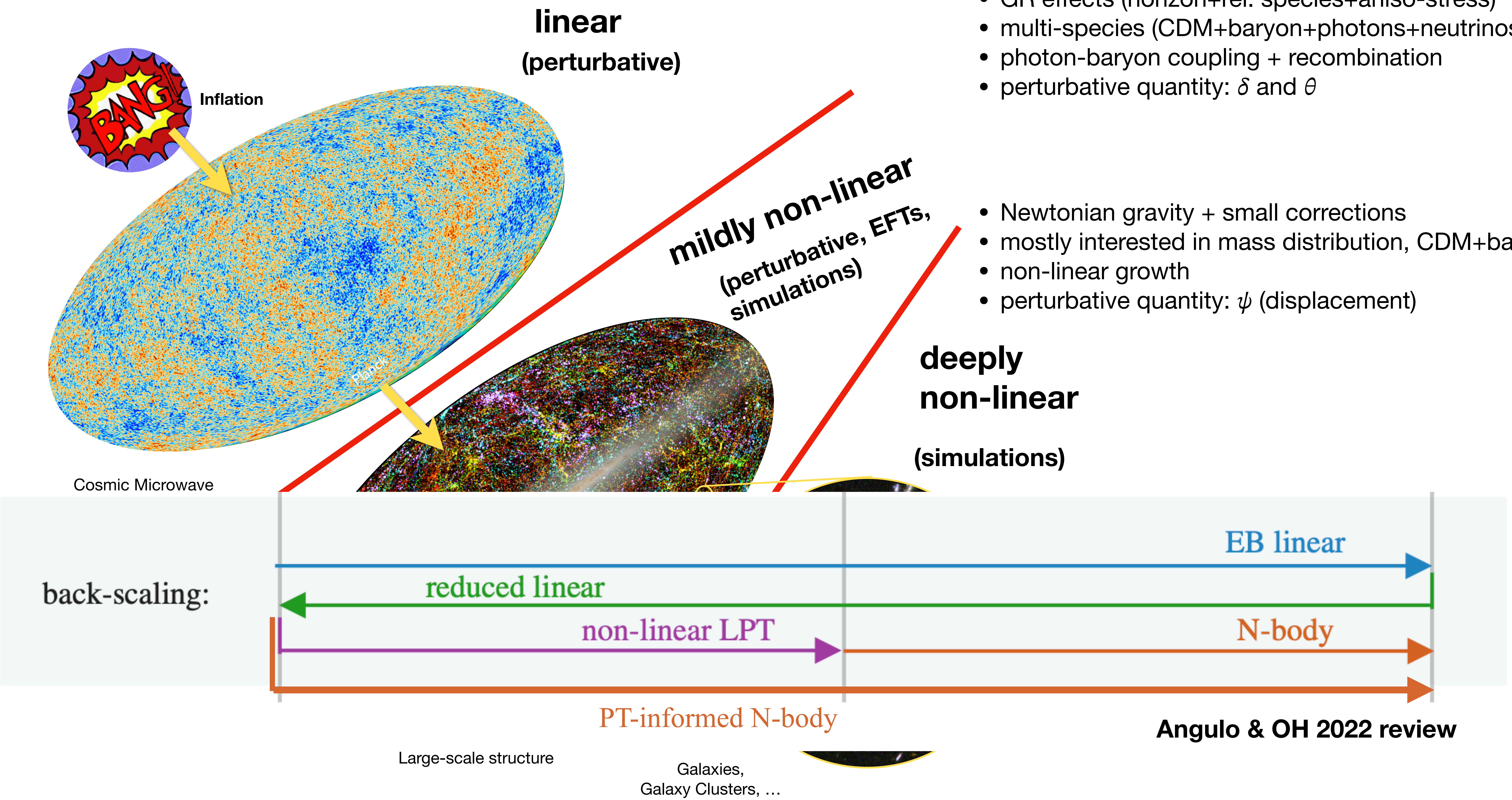


Angulo & OH 2022 review

Simulating Gaussian Random Fields (GRFs) – black board

Simulating Gaussian Random Fields (GRFs) – jupyter notebook

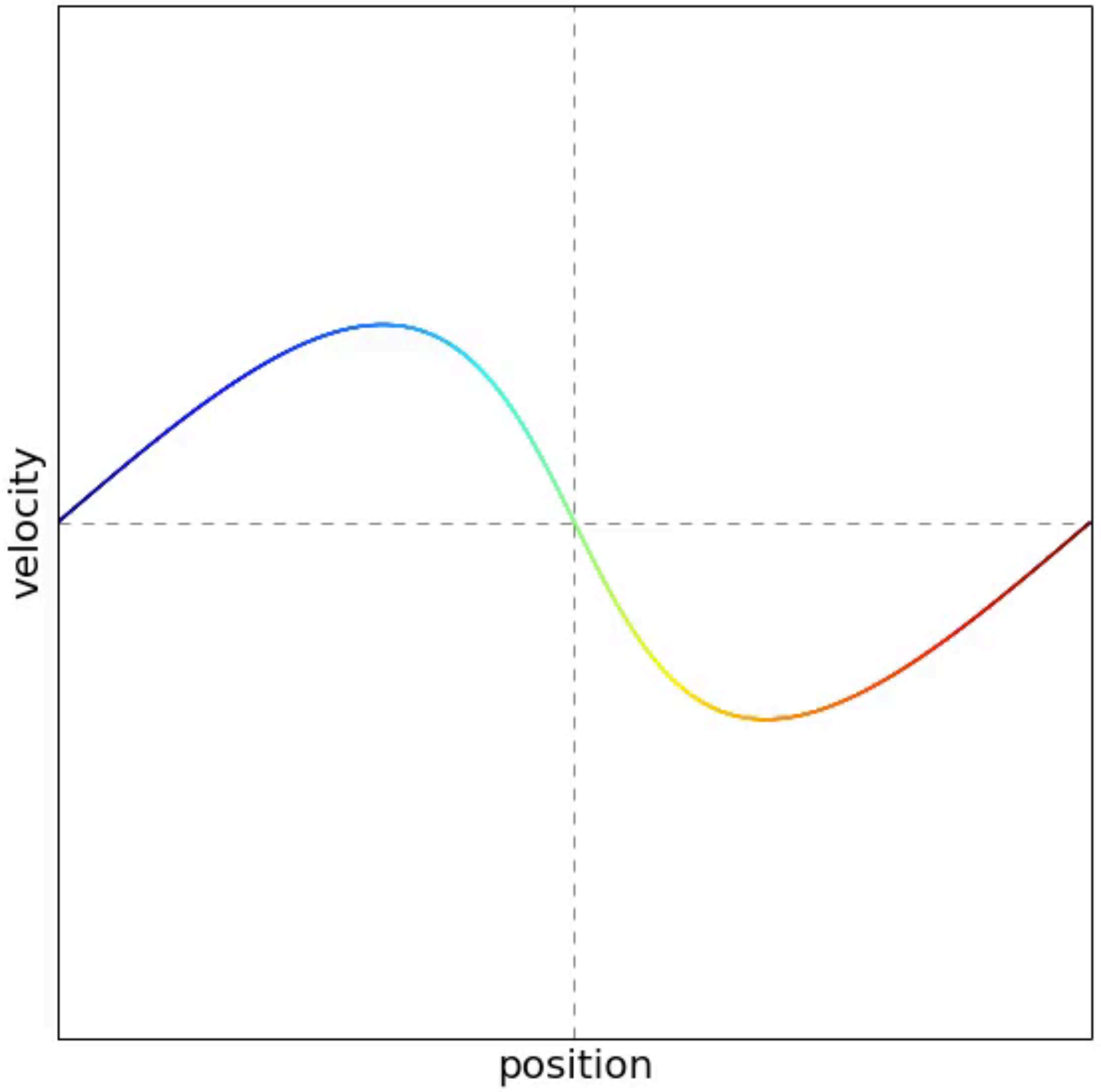
Lagrangian Perturbation Theory
(as initial conditions)

Non-Linear Evolution of Fluctuations

Cold Dark Matter lives on Lagrangian submanifold

Solve Vlasov-Poisson on (submanifold) characteristics $(\boldsymbol{q}, t) \mapsto (\boldsymbol{X}_{\boldsymbol{q}}(t), \boldsymbol{V}_{\boldsymbol{q}}(t)),$

$$\frac{\partial f}{\partial t} + \frac{\boldsymbol{v}}{a^2} \cdot \boldsymbol{\nabla}_x f - \boldsymbol{\nabla}_x \phi \cdot \boldsymbol{\nabla}_v f = 0 \qquad \Leftrightarrow \qquad \boldsymbol{X}_{\boldsymbol{q}}'' + \mathcal{H} \boldsymbol{X}_{\boldsymbol{q}}' = -\boldsymbol{\nabla} \phi(\boldsymbol{X}_{\boldsymbol{q}})$$



1D singularities: Rampf, Frisch & OH (2021)

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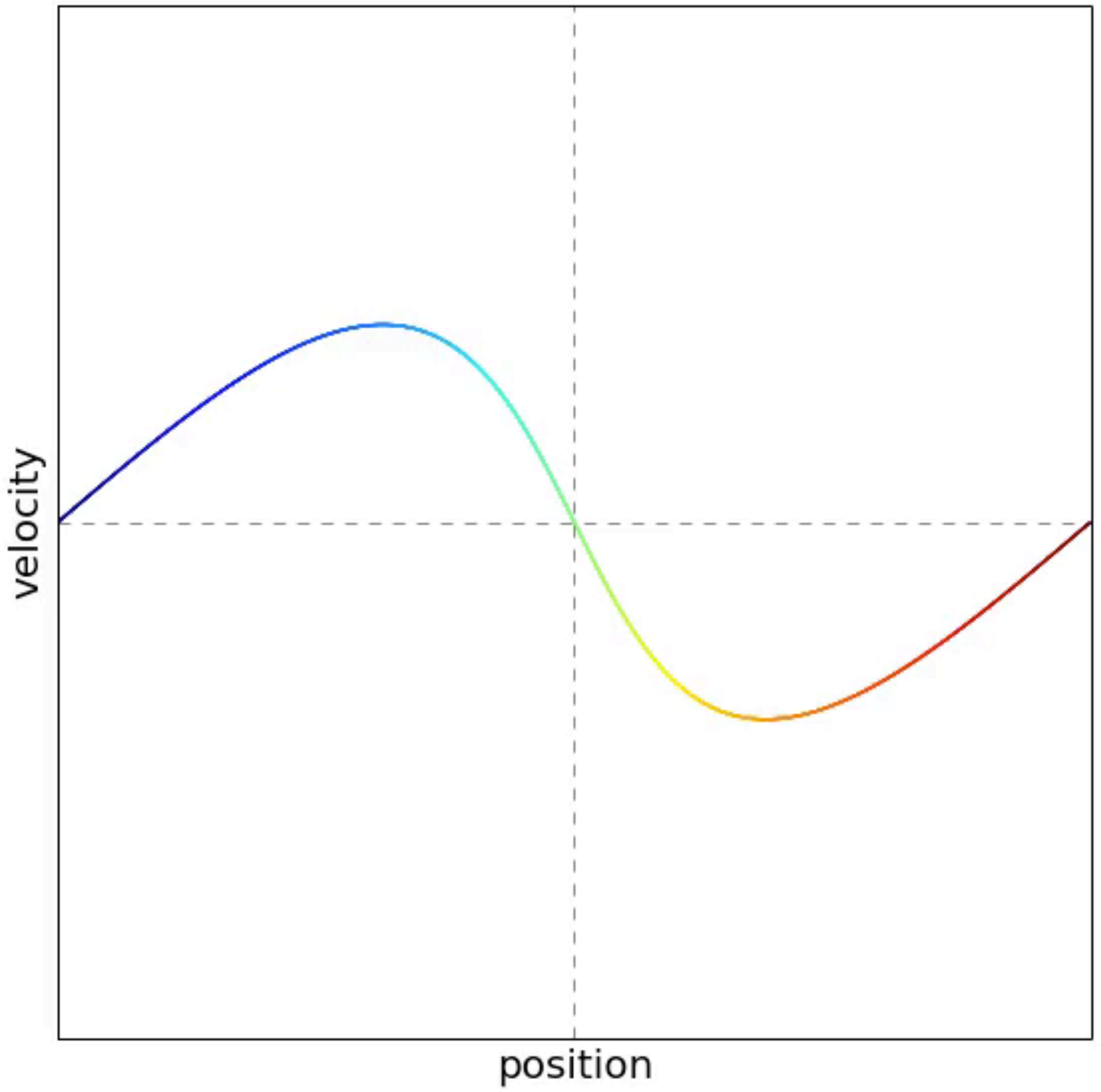

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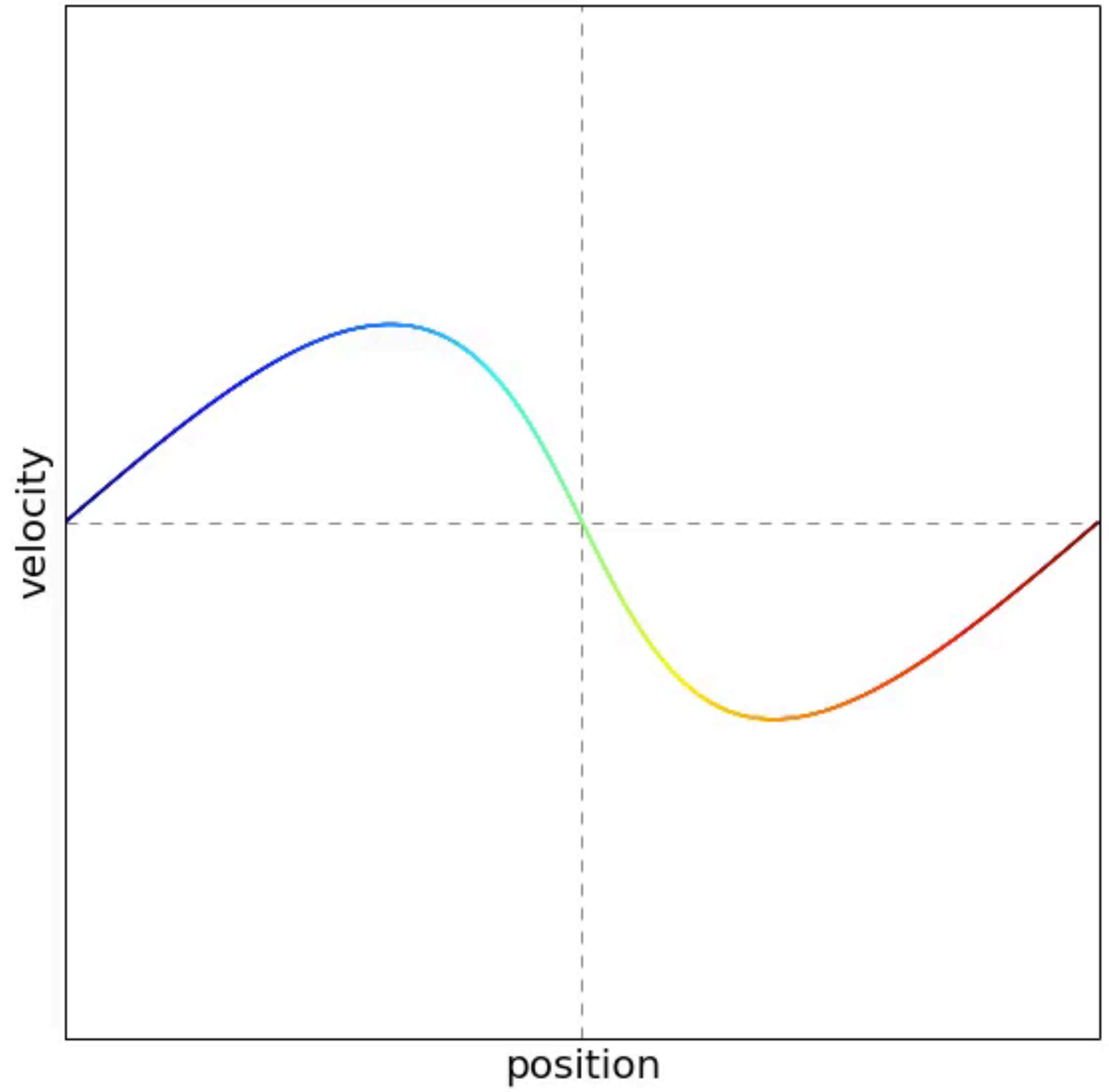
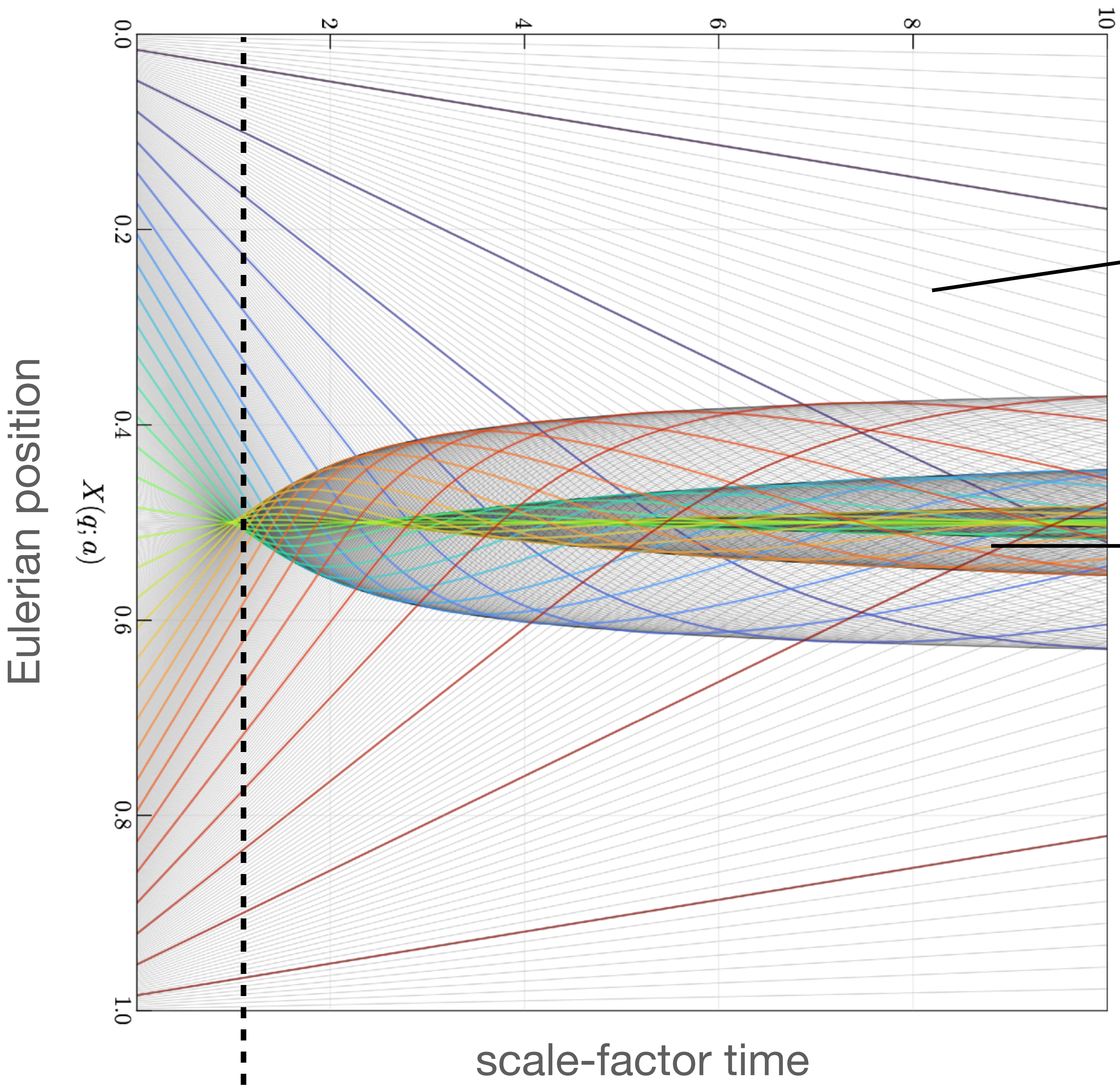
entering the multi-stream region is non-analytic (only finitely many bounded derivatives)



1D singularities: Rampf, Frisch & OH (2021)

density + acceleration singularities



moment of shell-crossing

monokinetic, single-valued
(analytic treatment possible)

multikinetic, multi-valued
(simulations, EFTs [by integrating out])

Zeldovich (1970) solution (straight lines) is exact prior to shell-crossing and outside shell-crossed regions

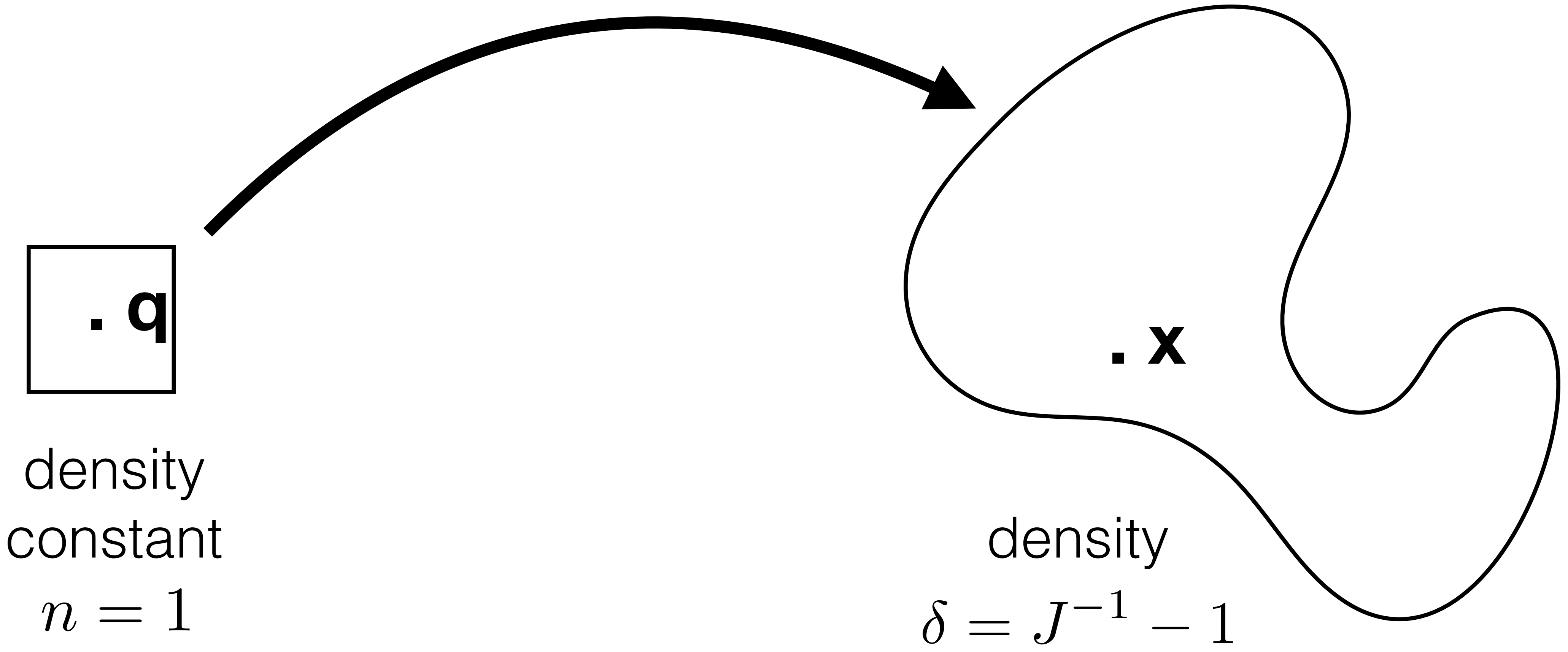
Lagrangian Perturbation Theory (LPT)
(for single fluid with cold initial data)

Lagrangian map

$$\boldsymbol{X}_{\boldsymbol{q}}(t) = \boldsymbol{q} + \boldsymbol{\Psi}(\boldsymbol{q}, t)$$

Overdensity given by Jacobian

$$\delta(\boldsymbol{x}, t) = \frac{1}{J(\boldsymbol{q}, t)} - 1$$



Shell-crossing: loss of bijectivity of Lagrangian map when J=0

Black board derivation ZA + LPT

Lagrangian Perturbation Theory
(black board derivation)

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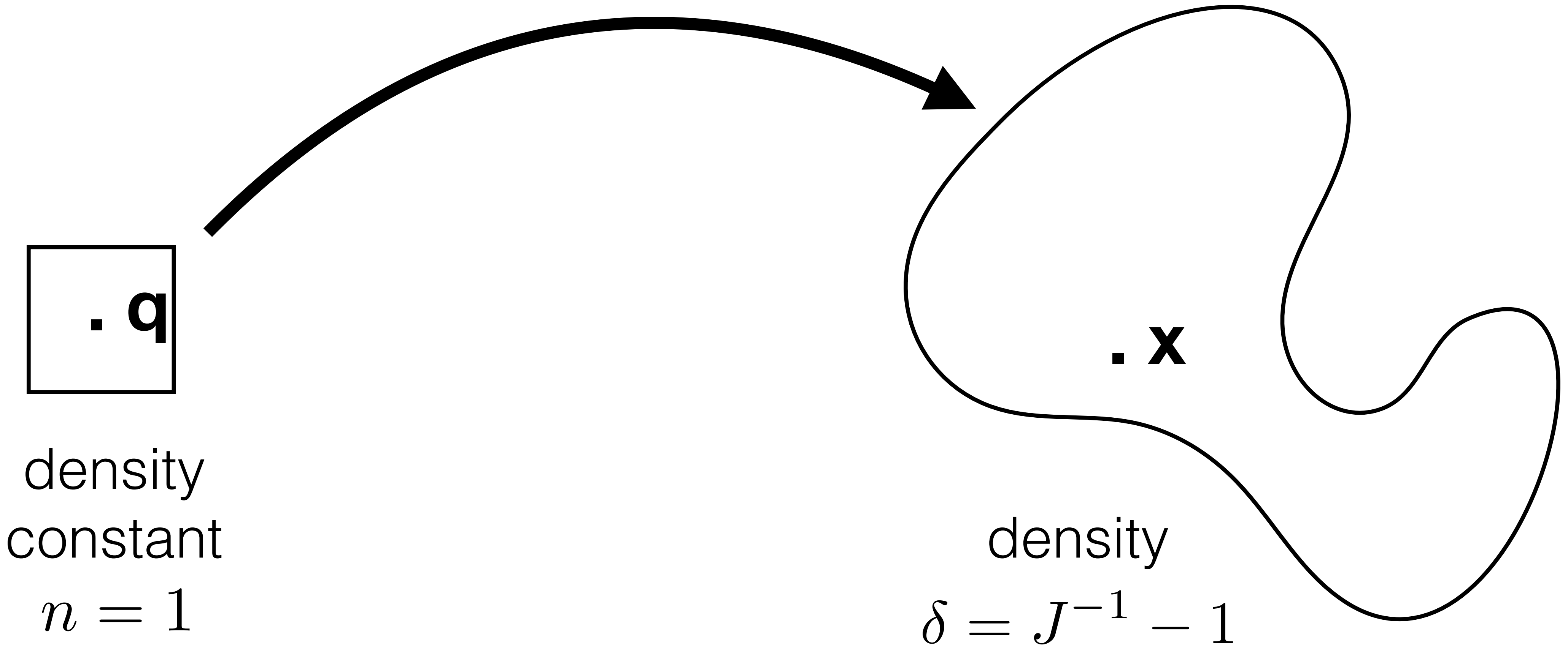

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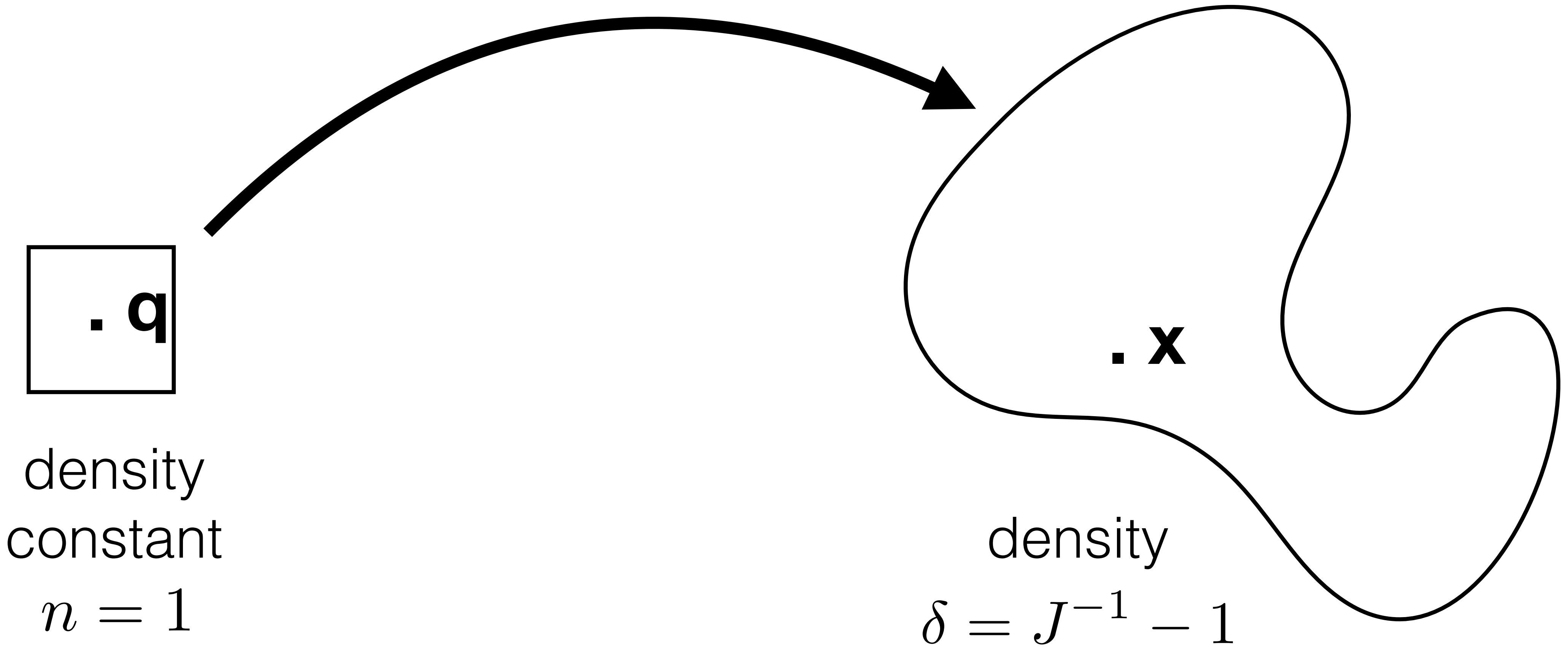

density
constant
$n = 1$

density
$\delta = J^{-1} - 1$

Solve as Taylor series (D is small parameter)

$$\boldsymbol{\Psi}(\mathbf{q}, \tau) = \sum_{n=1}^{\infty} D(\tau)^n \, \boldsymbol{\Psi}^{(n)}(\mathbf{q})$$

Buchert (1994), Catelan (1995), Bouchet+(1995), n=3
Rampf (2012), Zheligovsky&Frisch (2014), Matsubara (2015), all order

yields recursion relations to all orders.

For LCDM, actually $D^{(n)}(\tau) \neq D^n(\tau)$**,** see Rampf, Schobesberger & OH(2022), Fasiello, Fujita, Vlah (2022)

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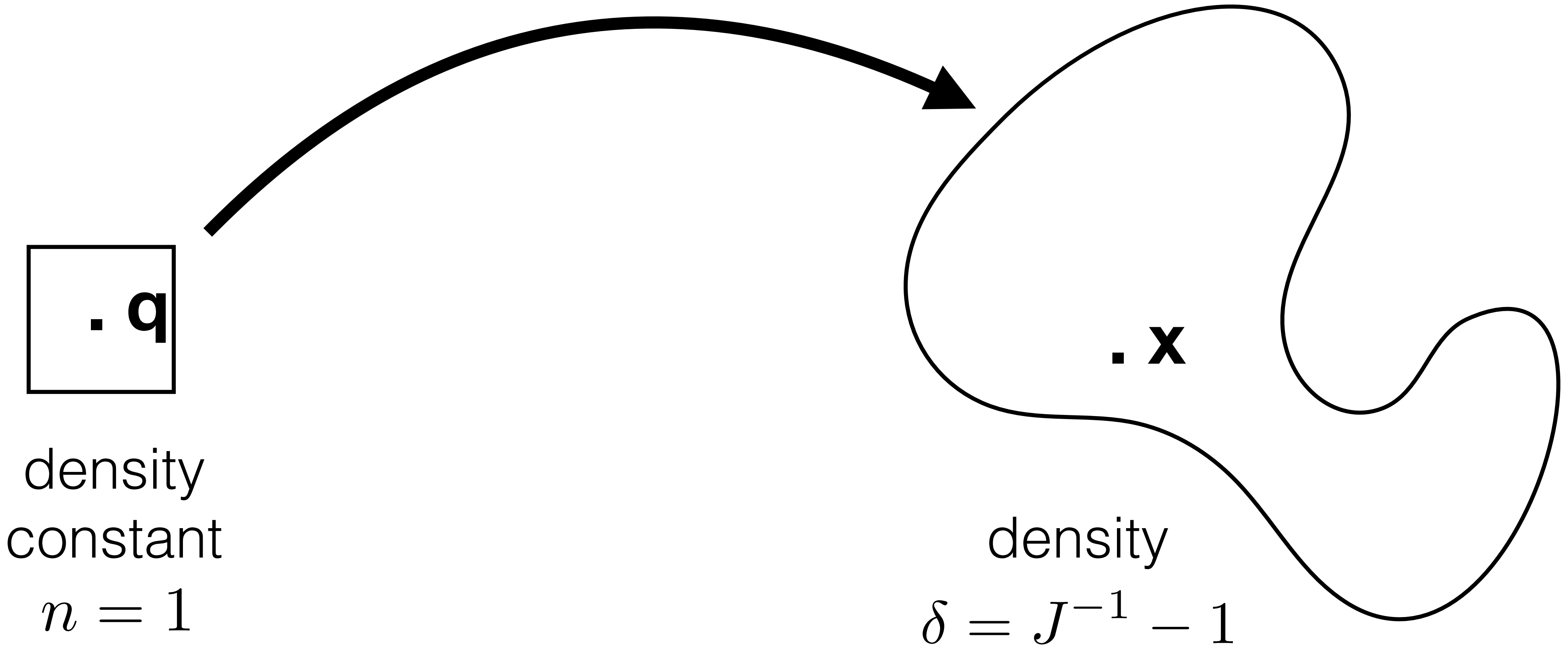

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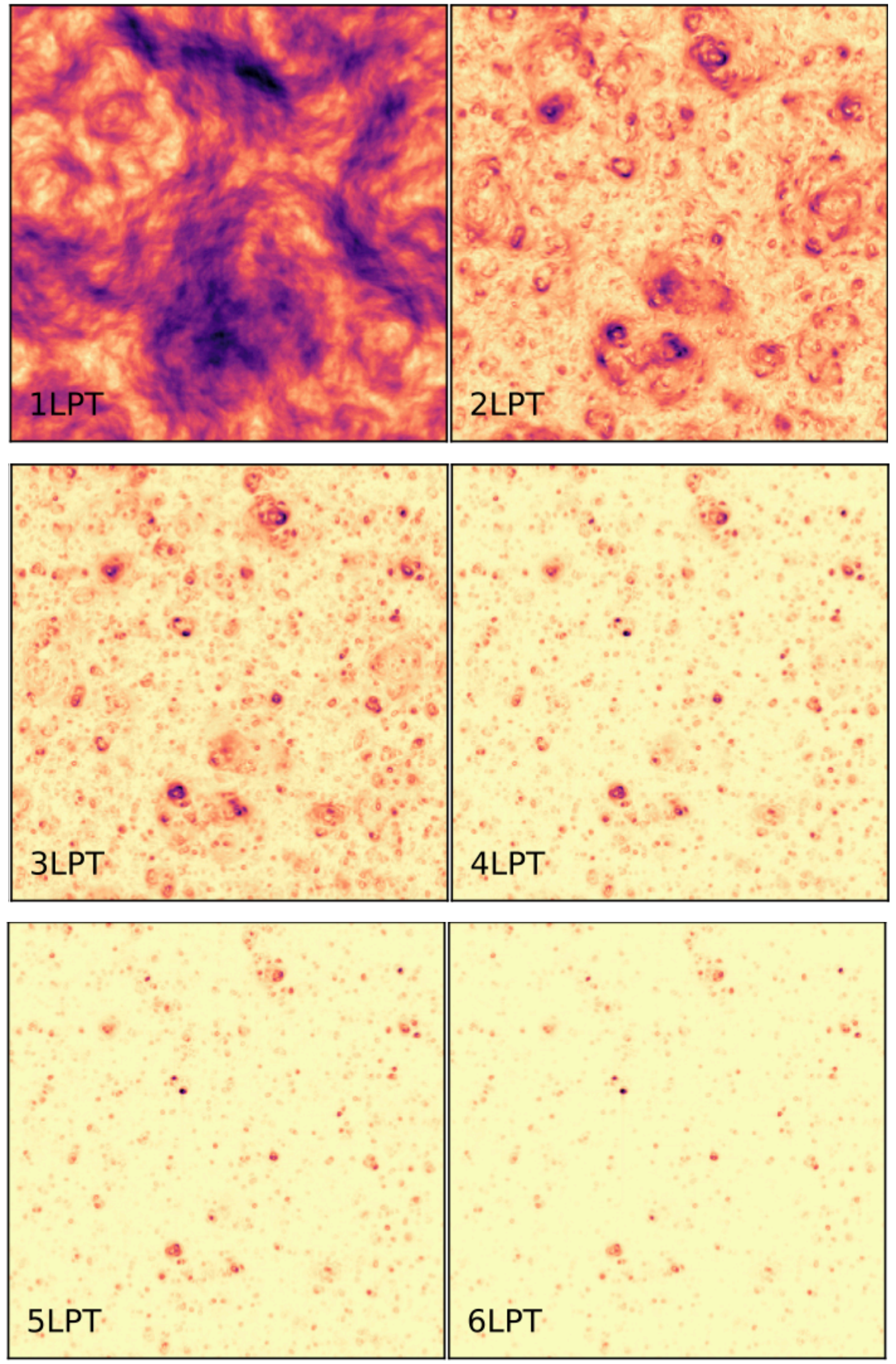

Rampf+OH (2021)

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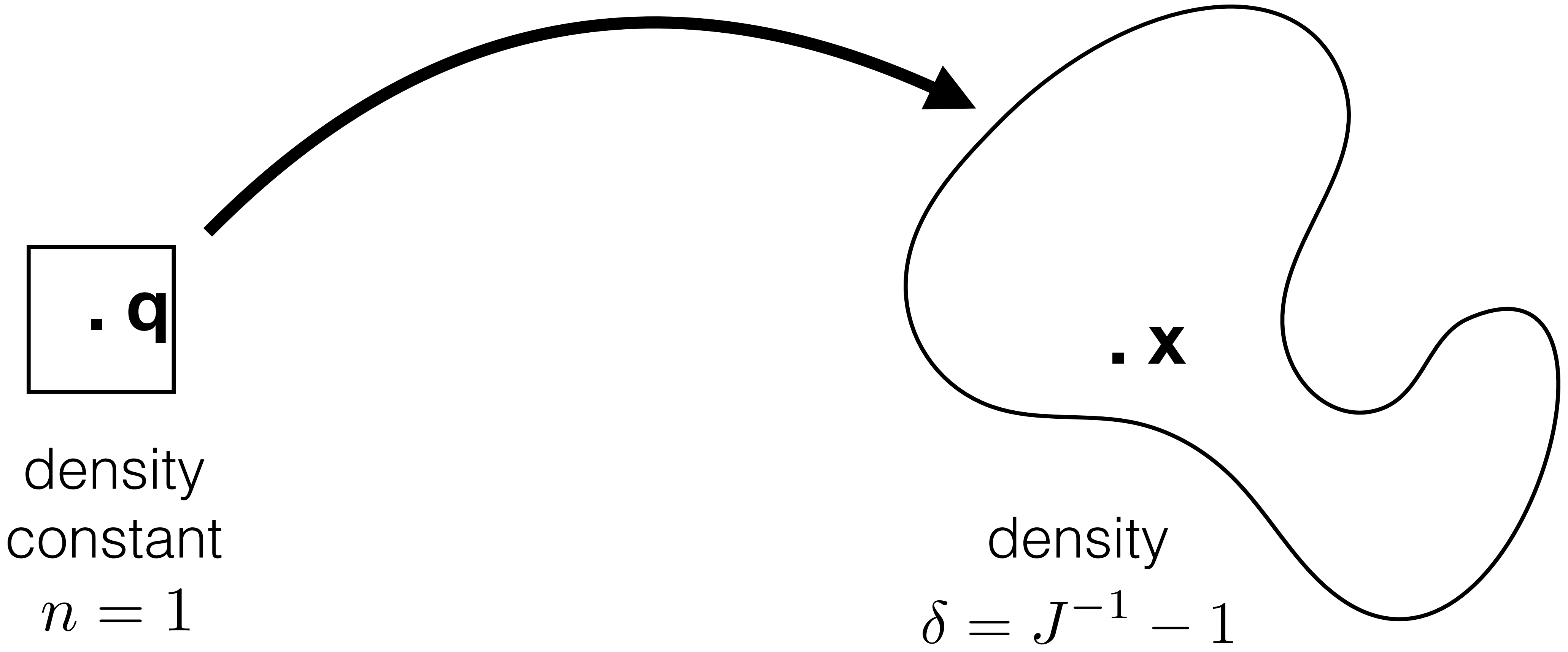

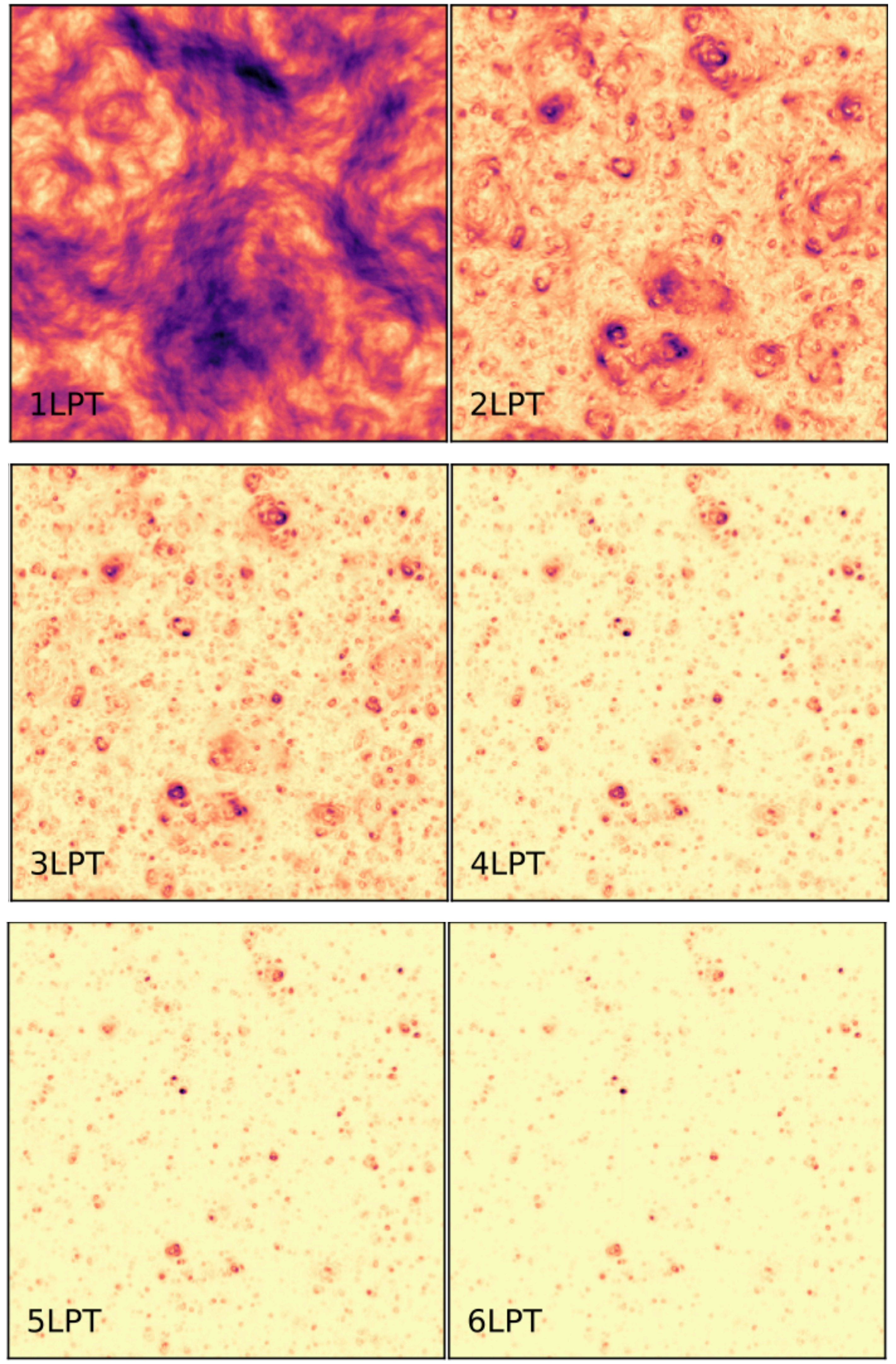

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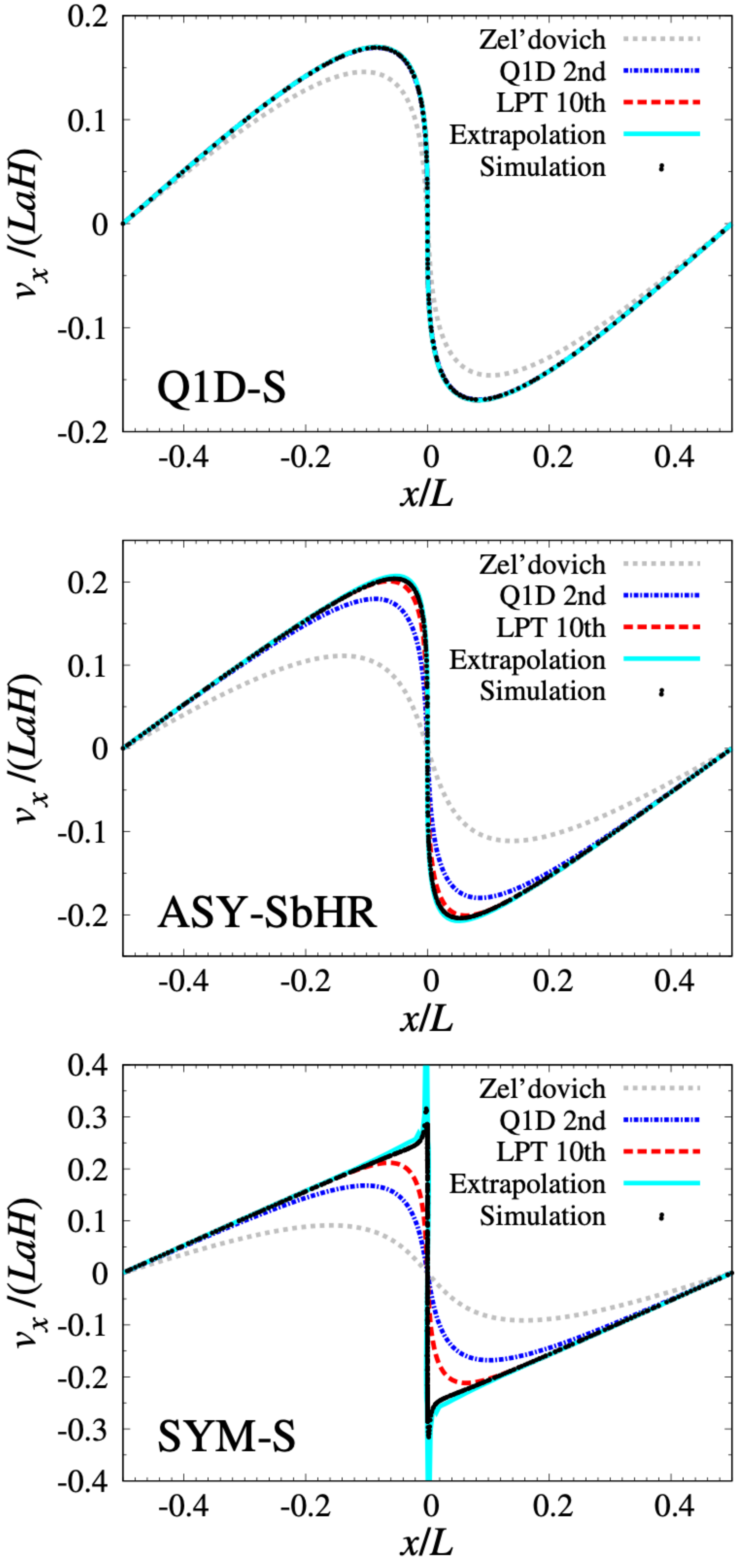
Convergence radius limited by 'shell-crossing', but not only!

Convergence limiting singularities



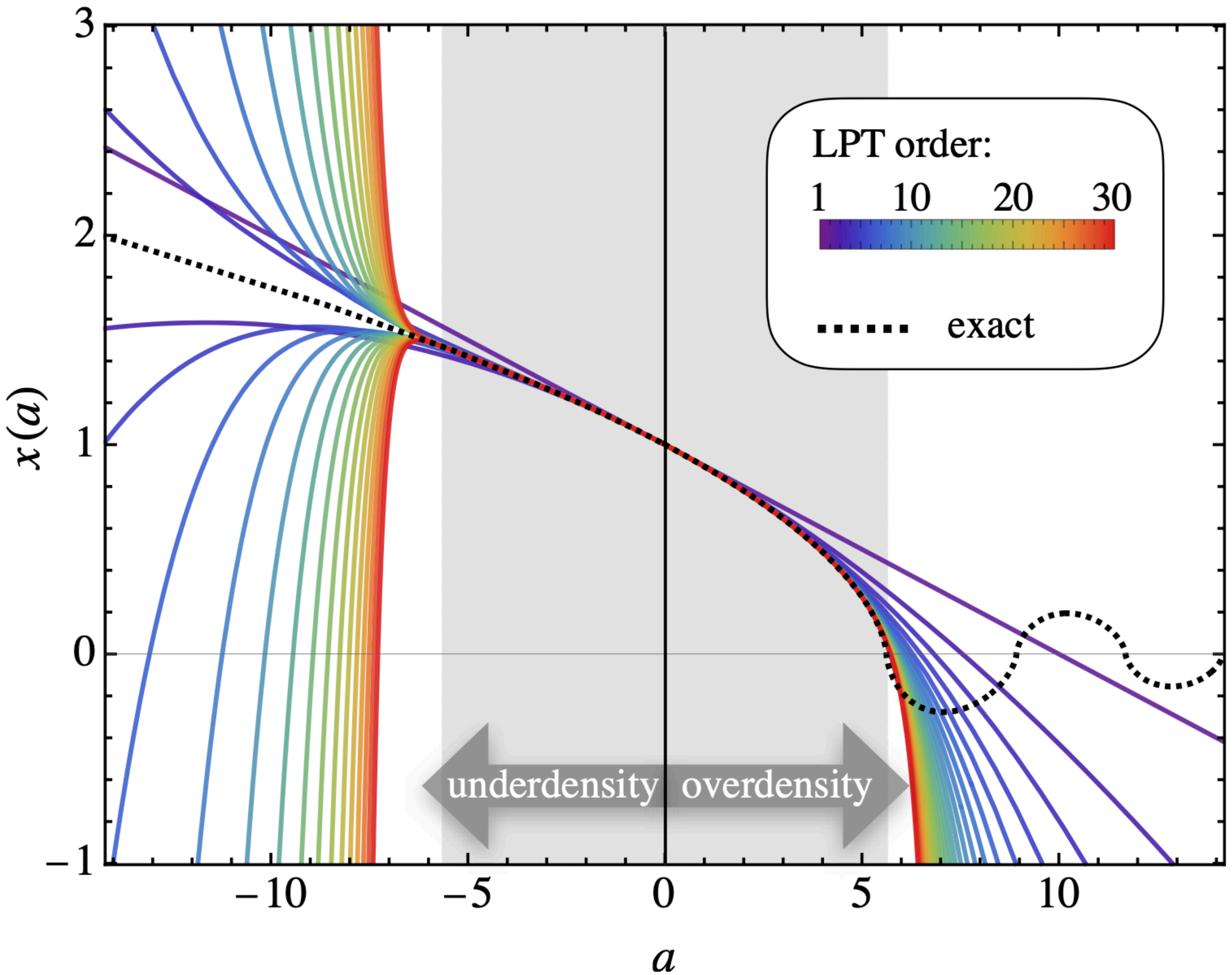
Saga, Colombi, Taruya 2019

Rampf, Frisch, OH '21: For 1D planar initial data, generic non-analyticity appears in the displacement

$$\boldsymbol{\Psi}(\boldsymbol{q},a) \propto (a - a_*(\boldsymbol{q}))^{\frac{5}{2}}$$

for 1D spherically symmetric initial data, this is hardened to (Rampf & OH '23)

$$\boldsymbol{\Psi}(\boldsymbol{q},a) \propto (a - a_*(\boldsymbol{q}))^{\frac{2}{3}}$$



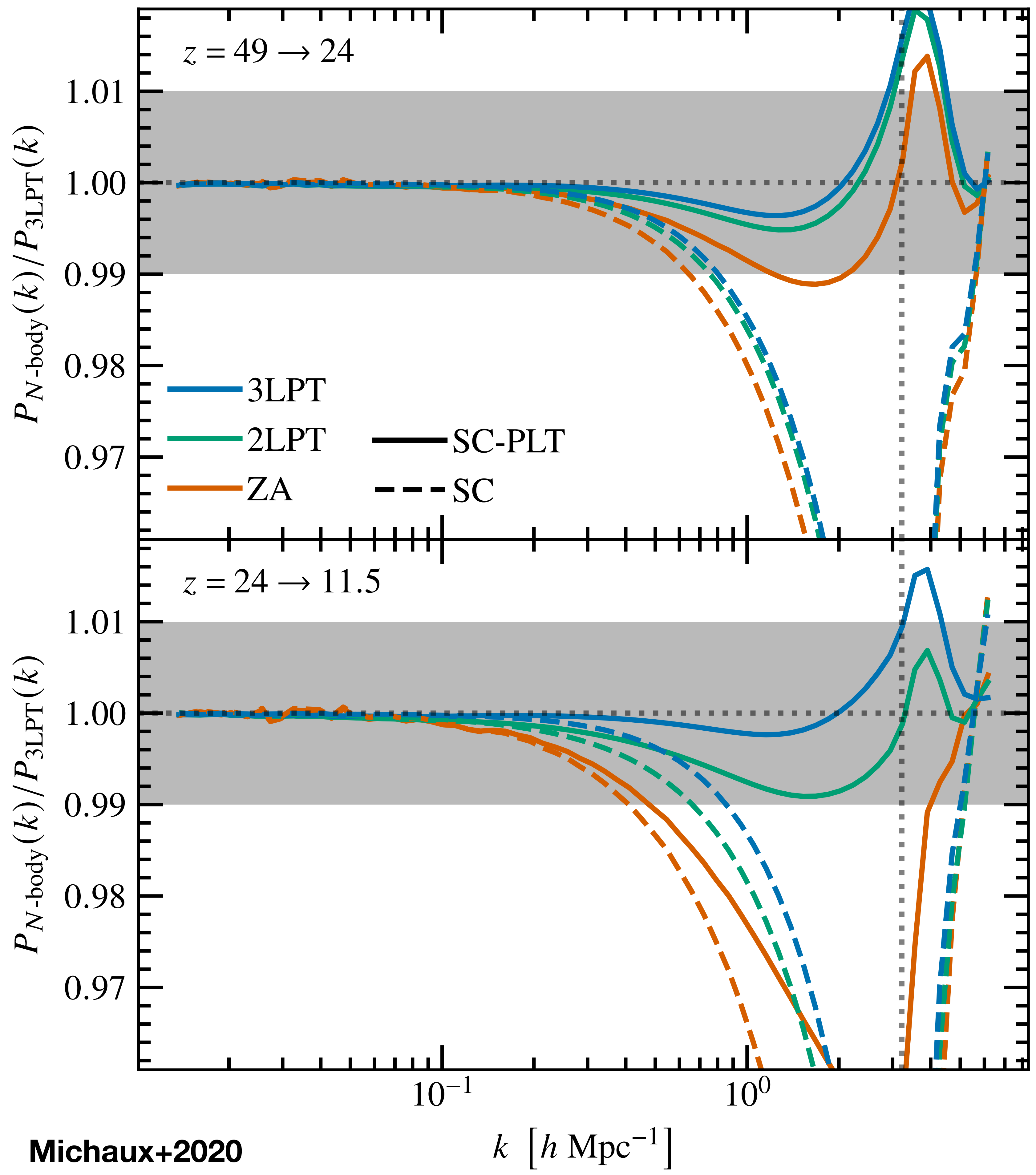
Rampf & OH 2022

Simulating ZA characteristcs from GRFs – jupyter notebook

Convergence between LPT and simulations

Agreement between N-body and LPT

Comparison of weakly evolved power spectra:



Michaux+2020

How is this possible? They model the same discrete set of modes

Agreement between N-body and LPT

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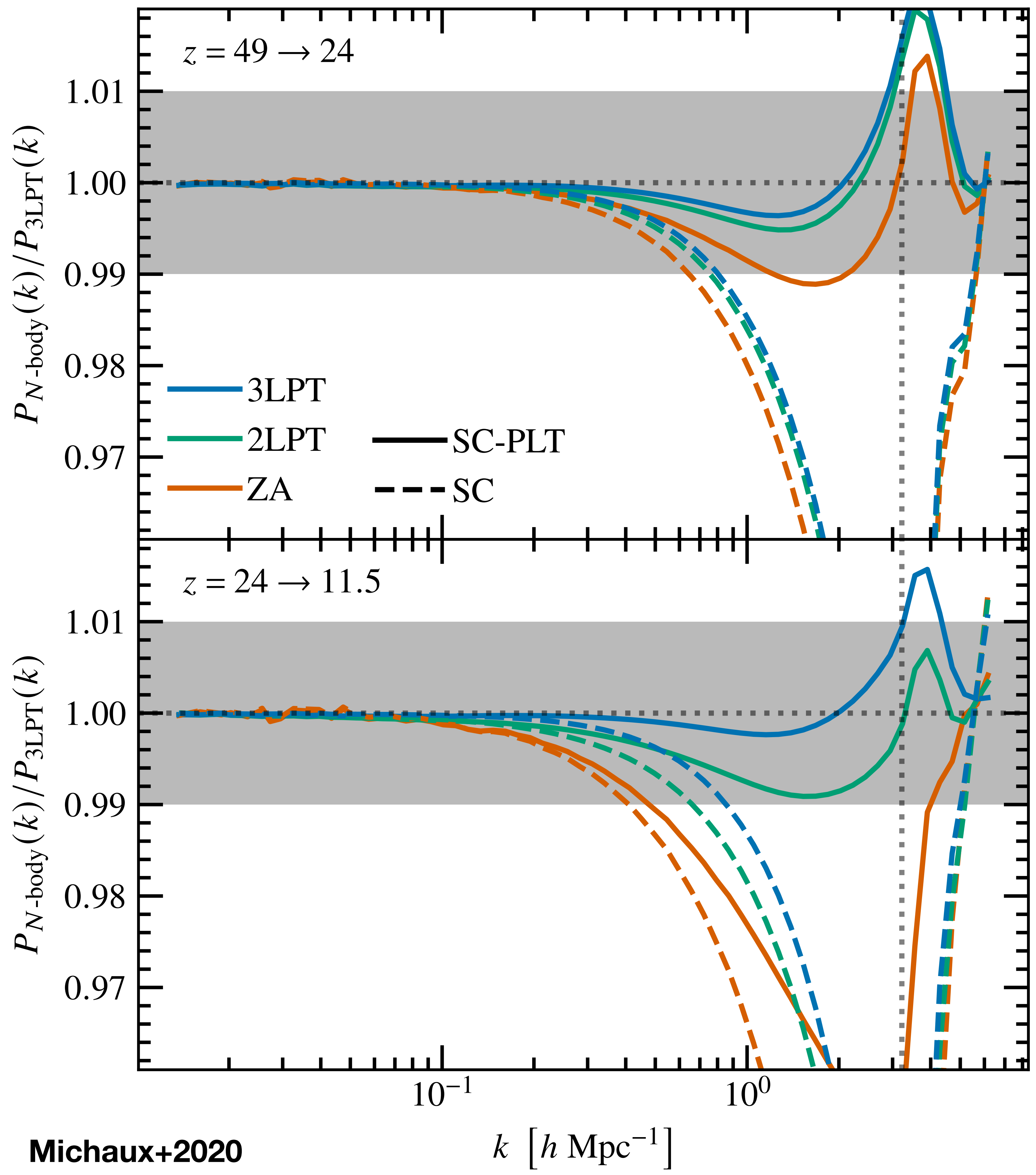

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Two sources of error:

1) the nLPT truncation error (Scoccimarro 1998, Crocce+2006) aka 'transients'

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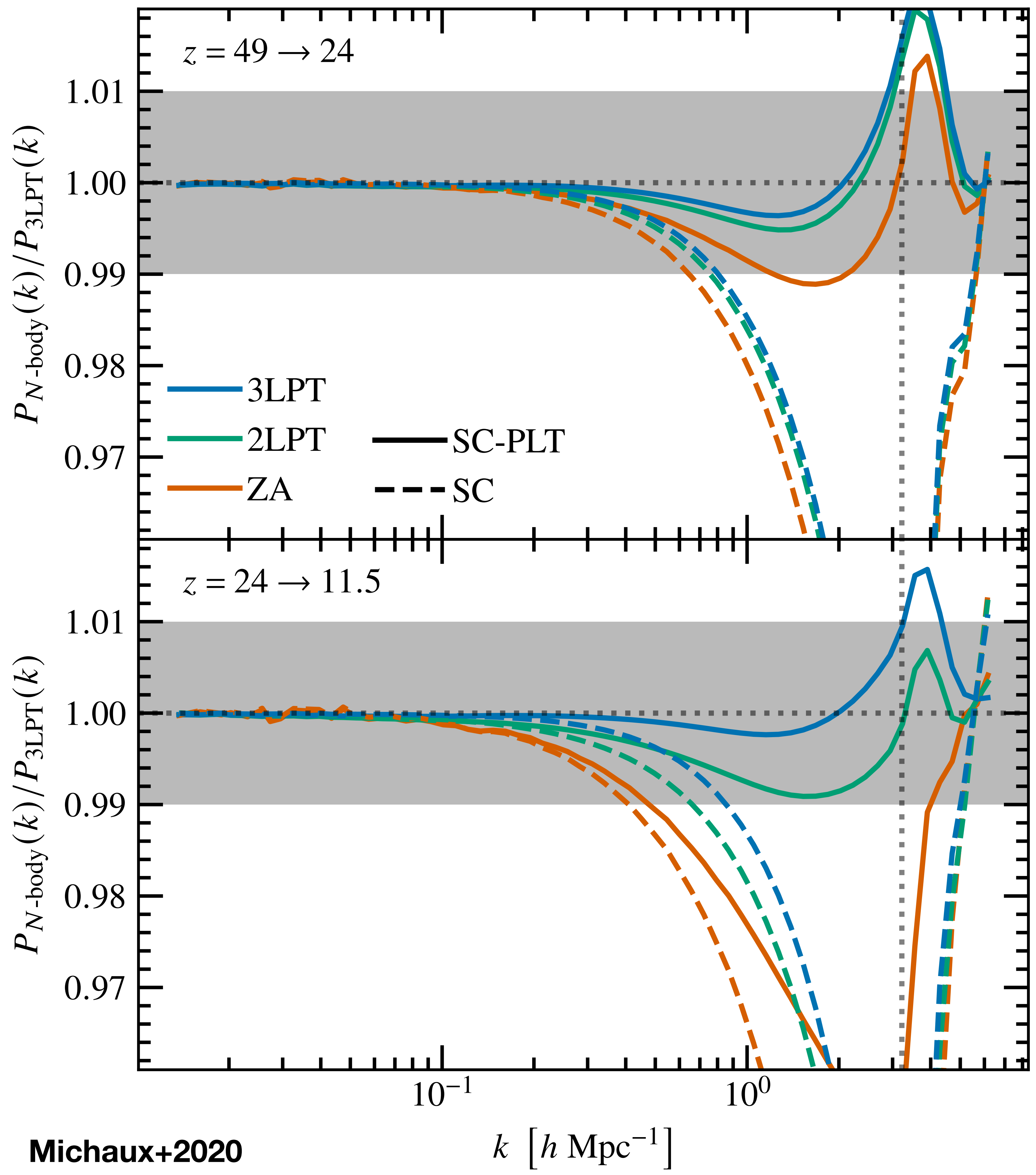
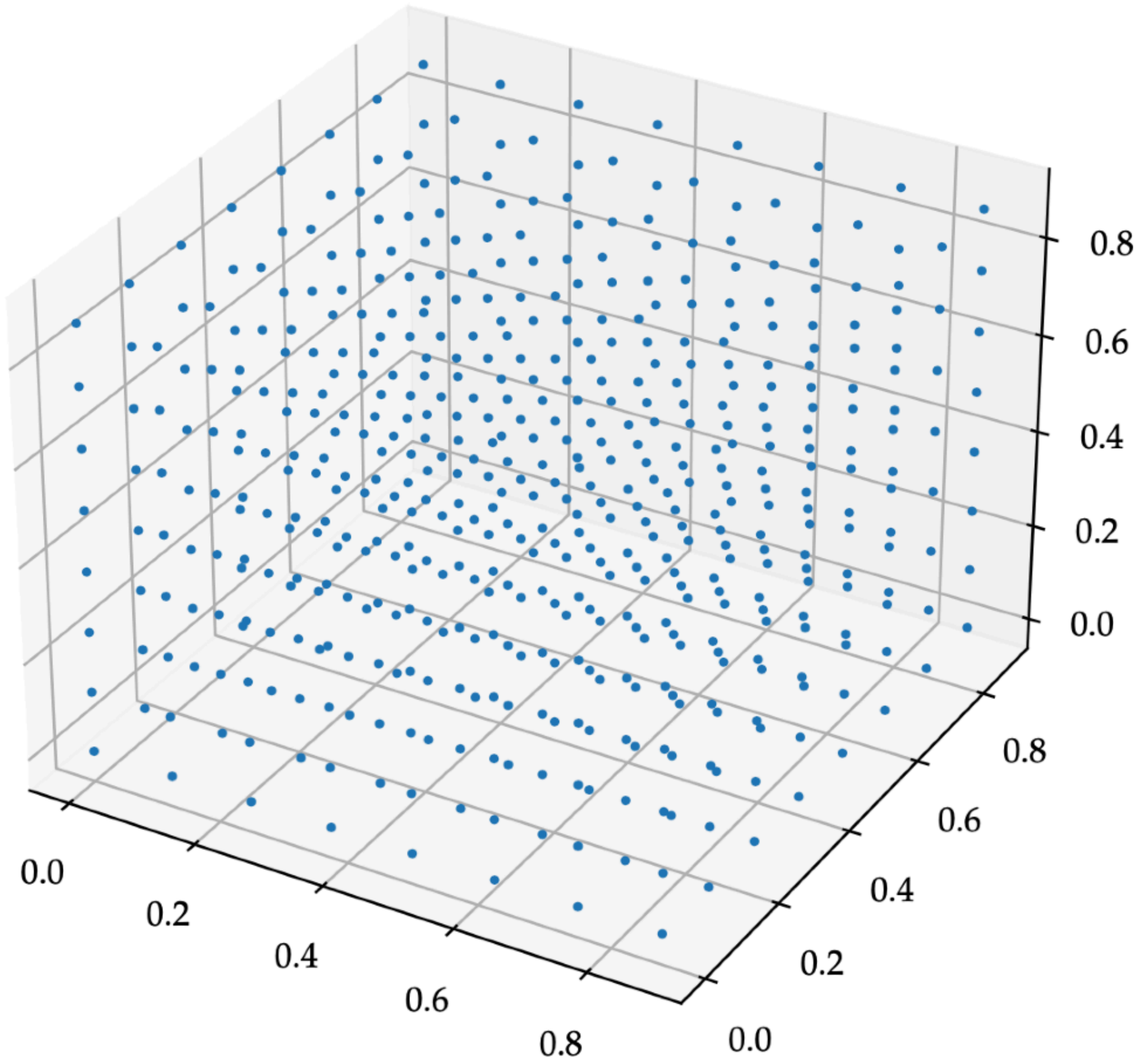


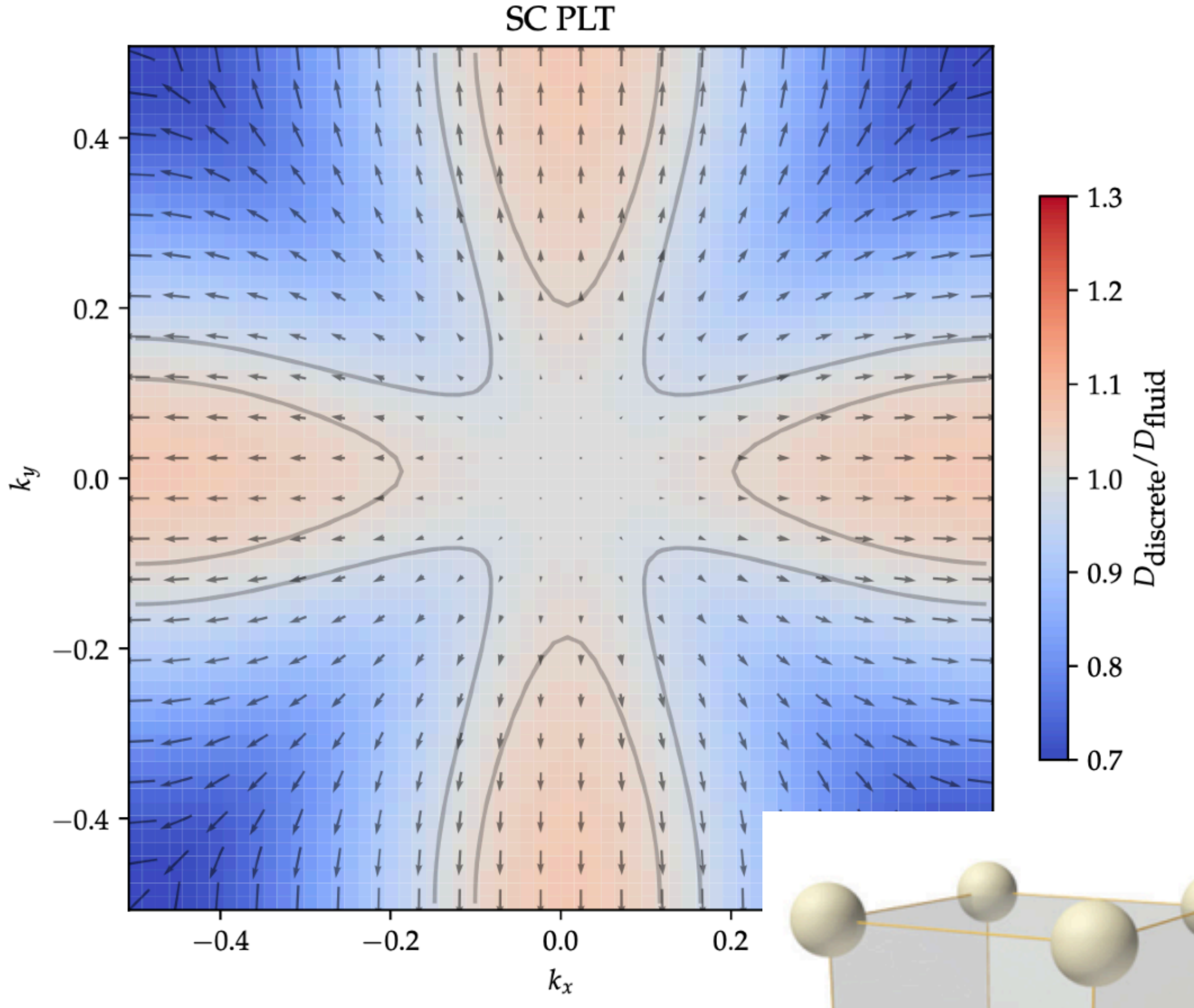
Michaux+2020

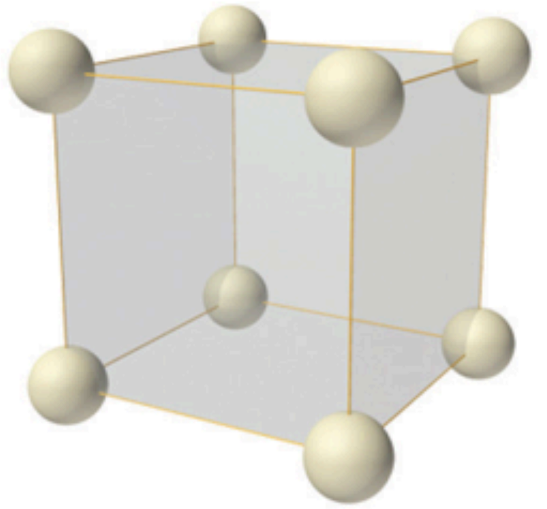
How is this possible? They model the same discrete set of modes

Two sources of error:

1) the nLPT truncation error (Scoccimarro 1998, Crocce+2006) aka 'transients'
2) the N-body discreteness (and force) errors:







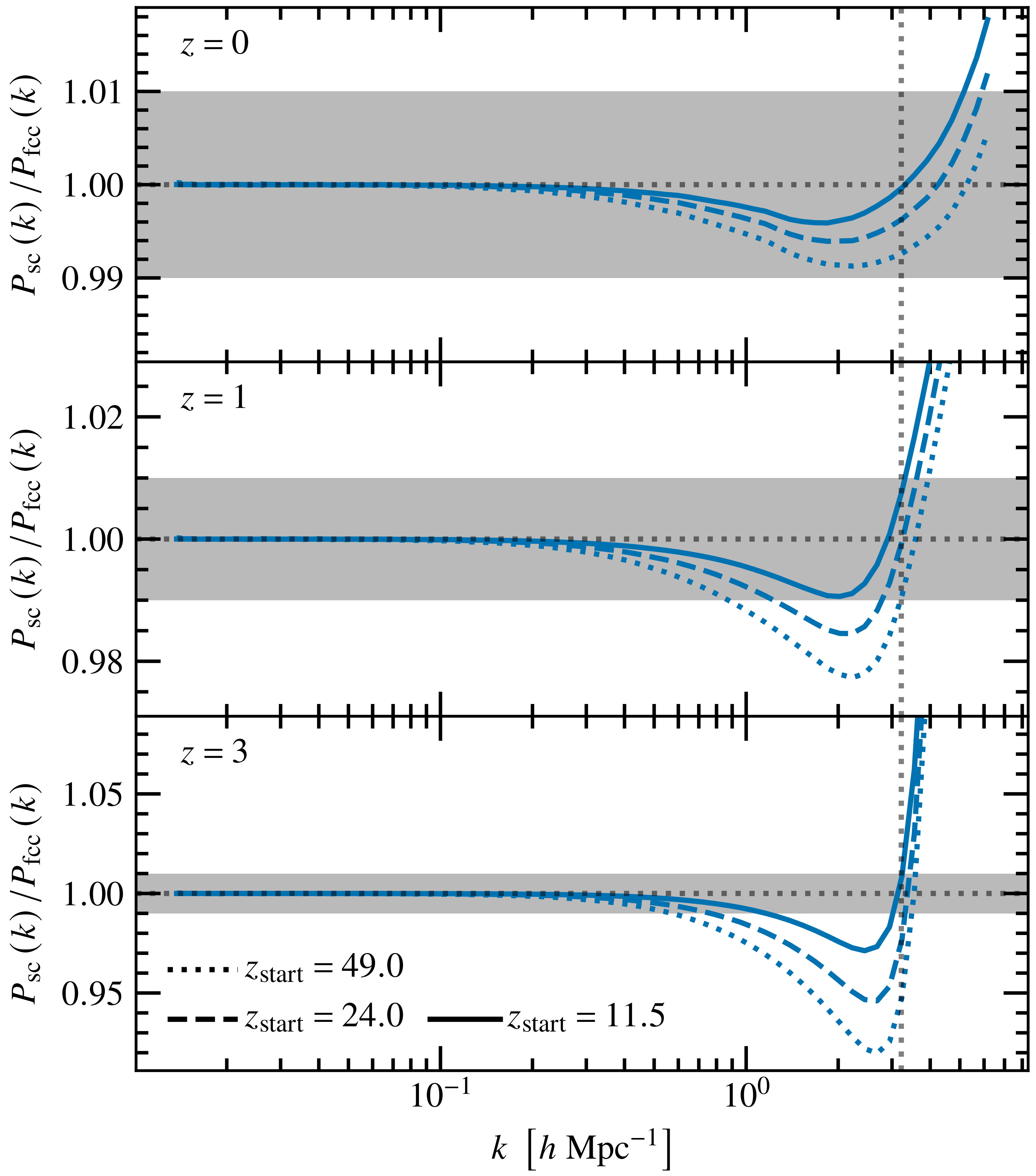
cf. Joyce+2005, Joyce&Marcos 2007, Marcos 2008, but also Garrison+2016

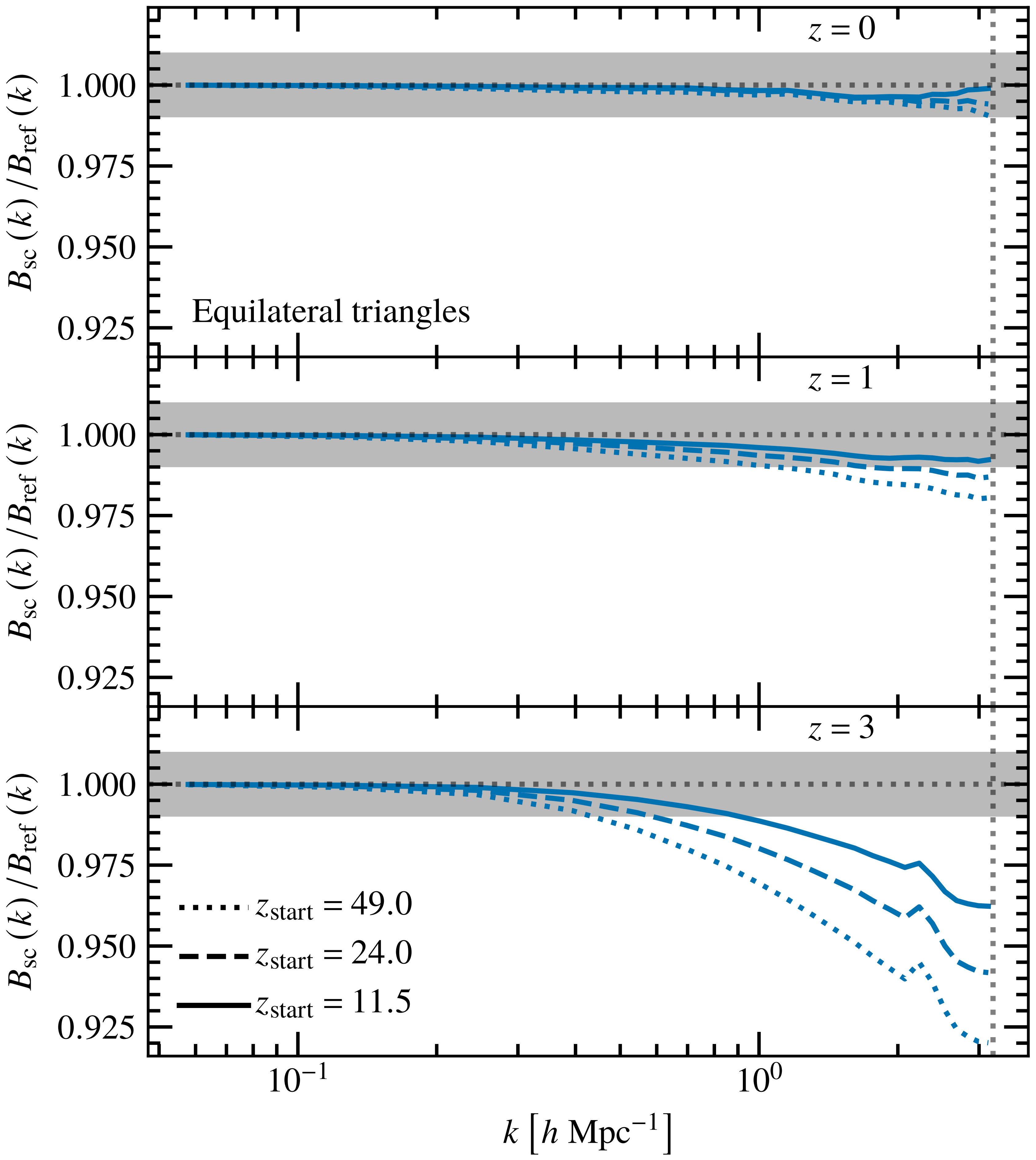
SC
(simple cubic)

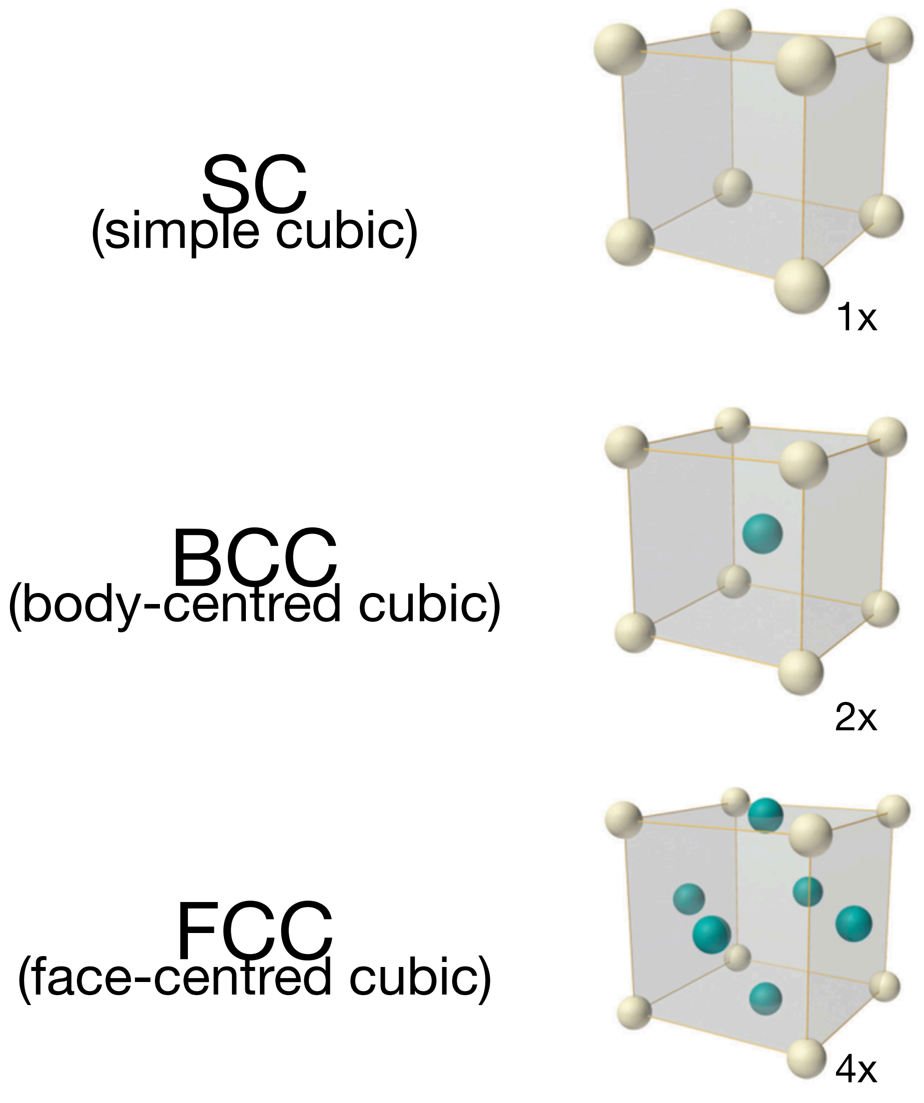
Discreteness — impact on low-z power spectrum

effect on PS at z=0 wiped out by non-linearity (scale-mixing), not at higher z

ref = FCC







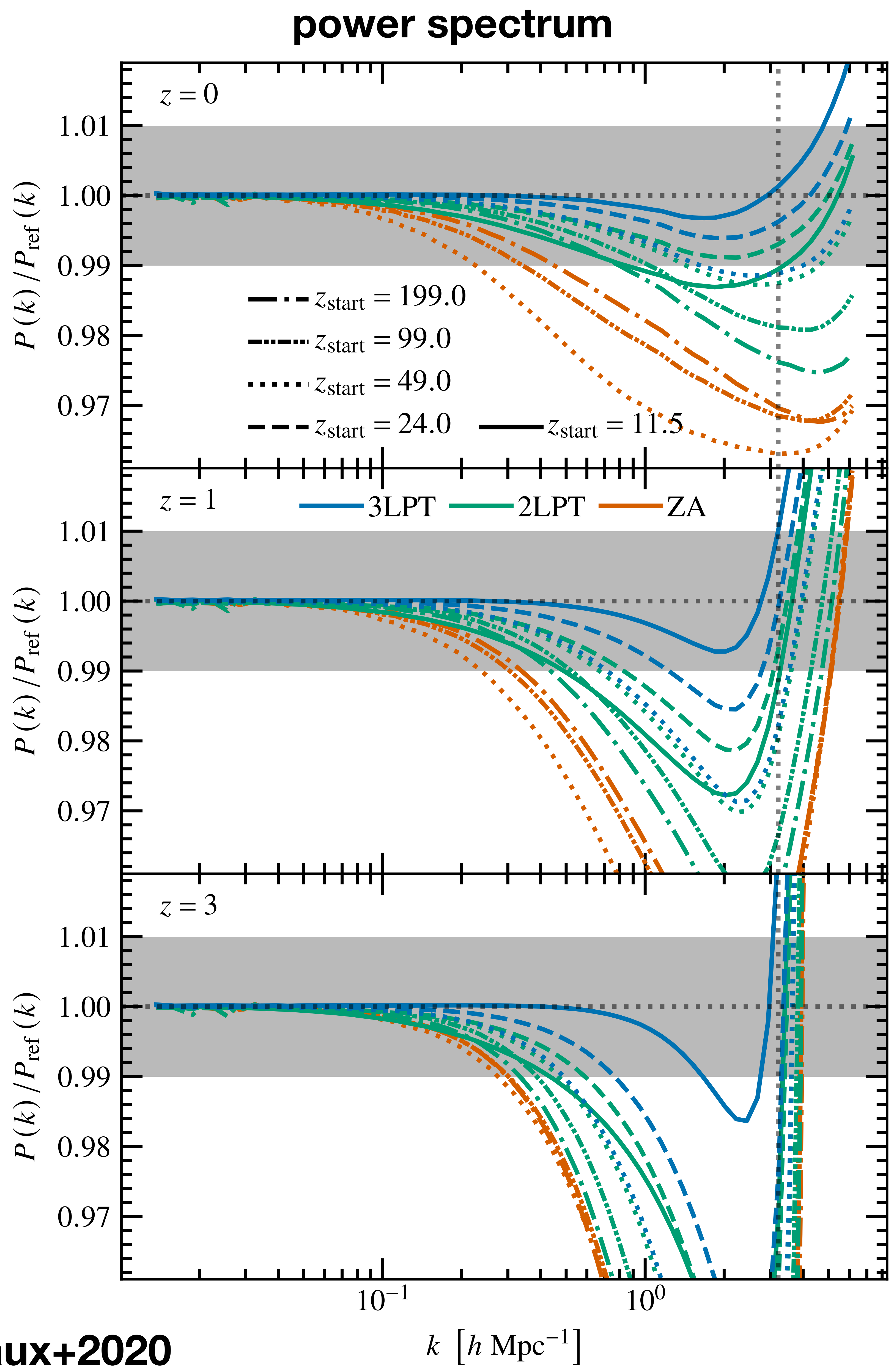
(cf. also Marcos 2008)

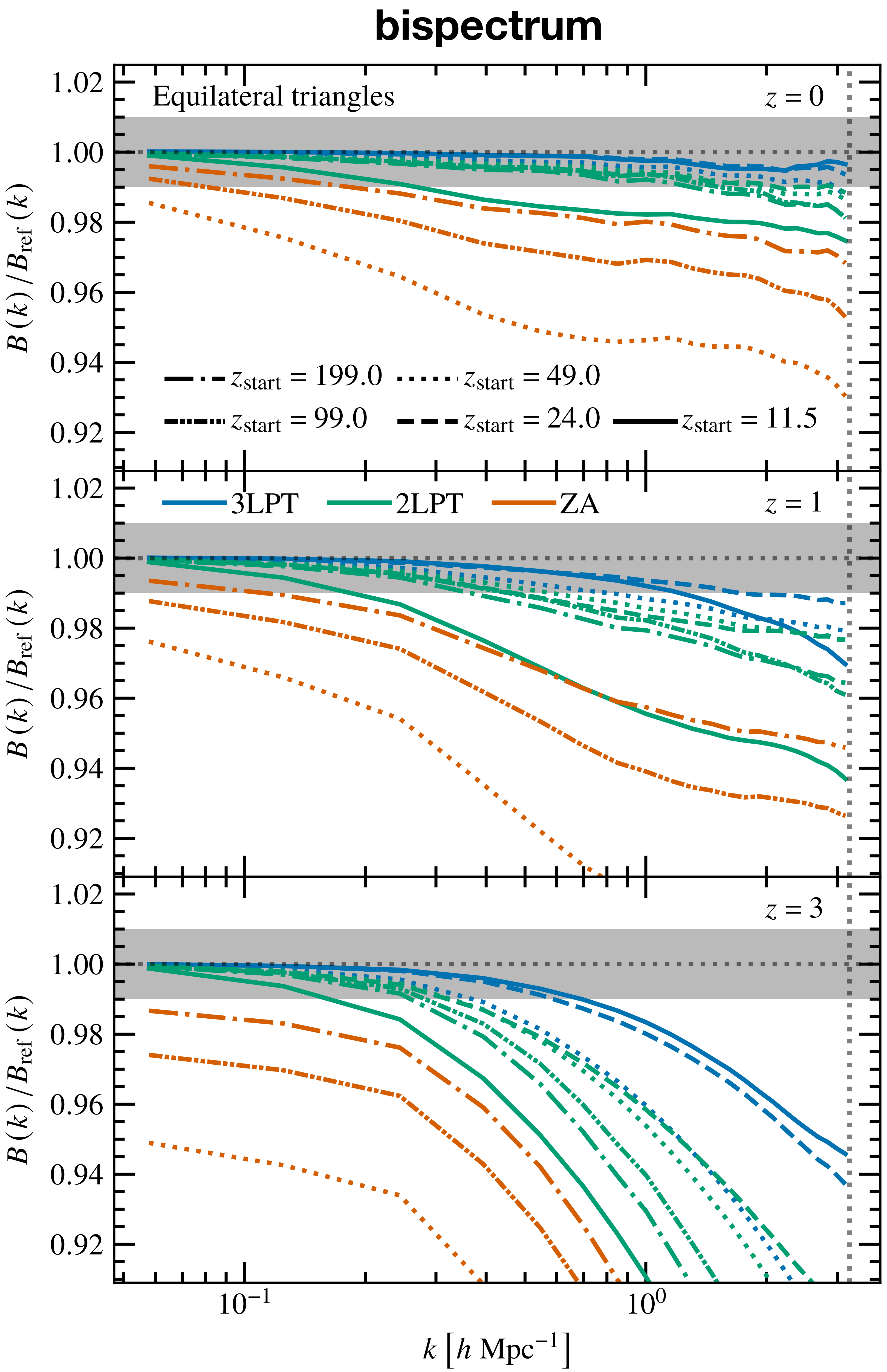
discreteness effects strongest at high z

very slow convergence with particle number ($\propto k_{\rm Ny}^3$)

Michaux+2020

Impact of nLPT vs. discreteness on low-z spectra



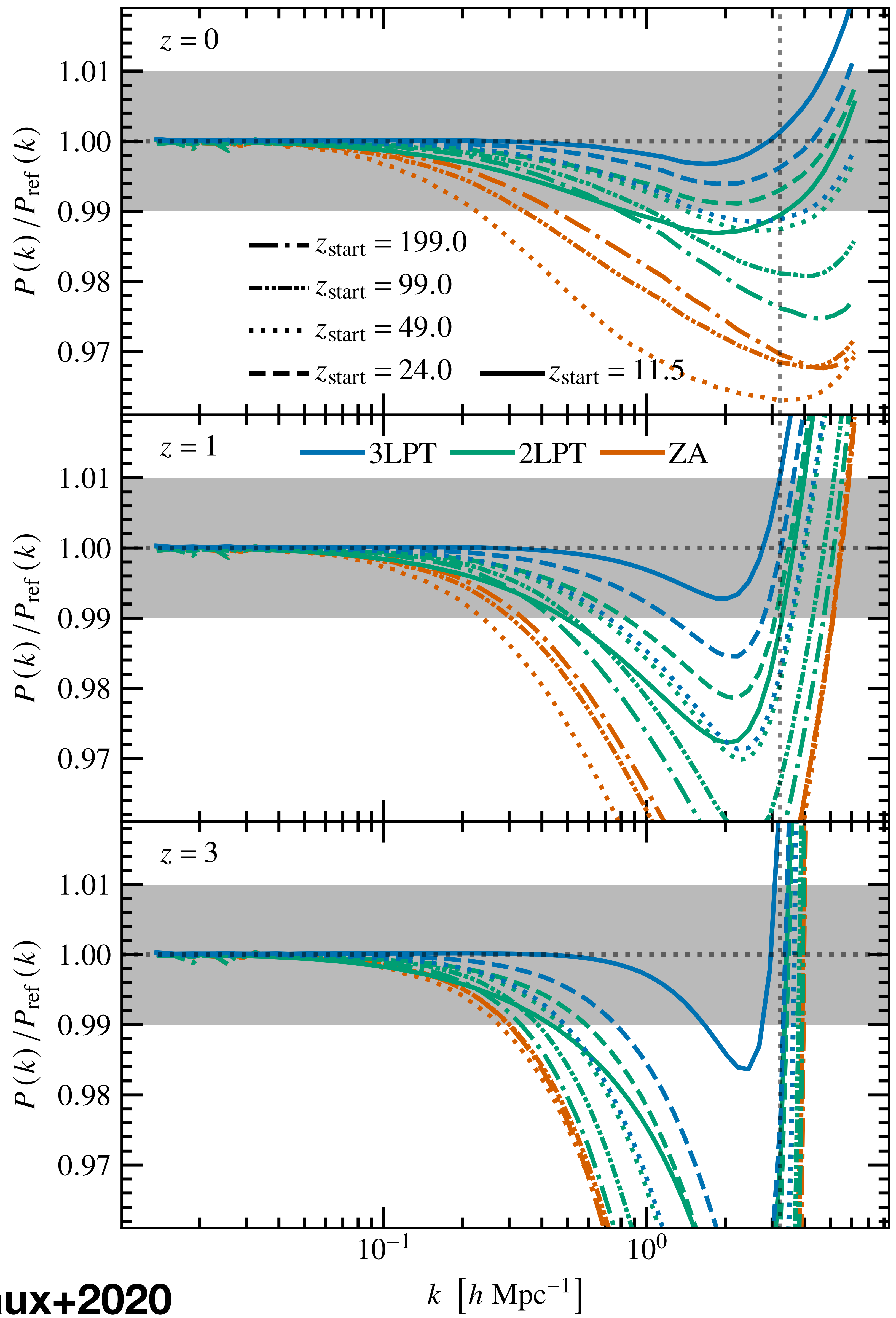


Michaux+2020

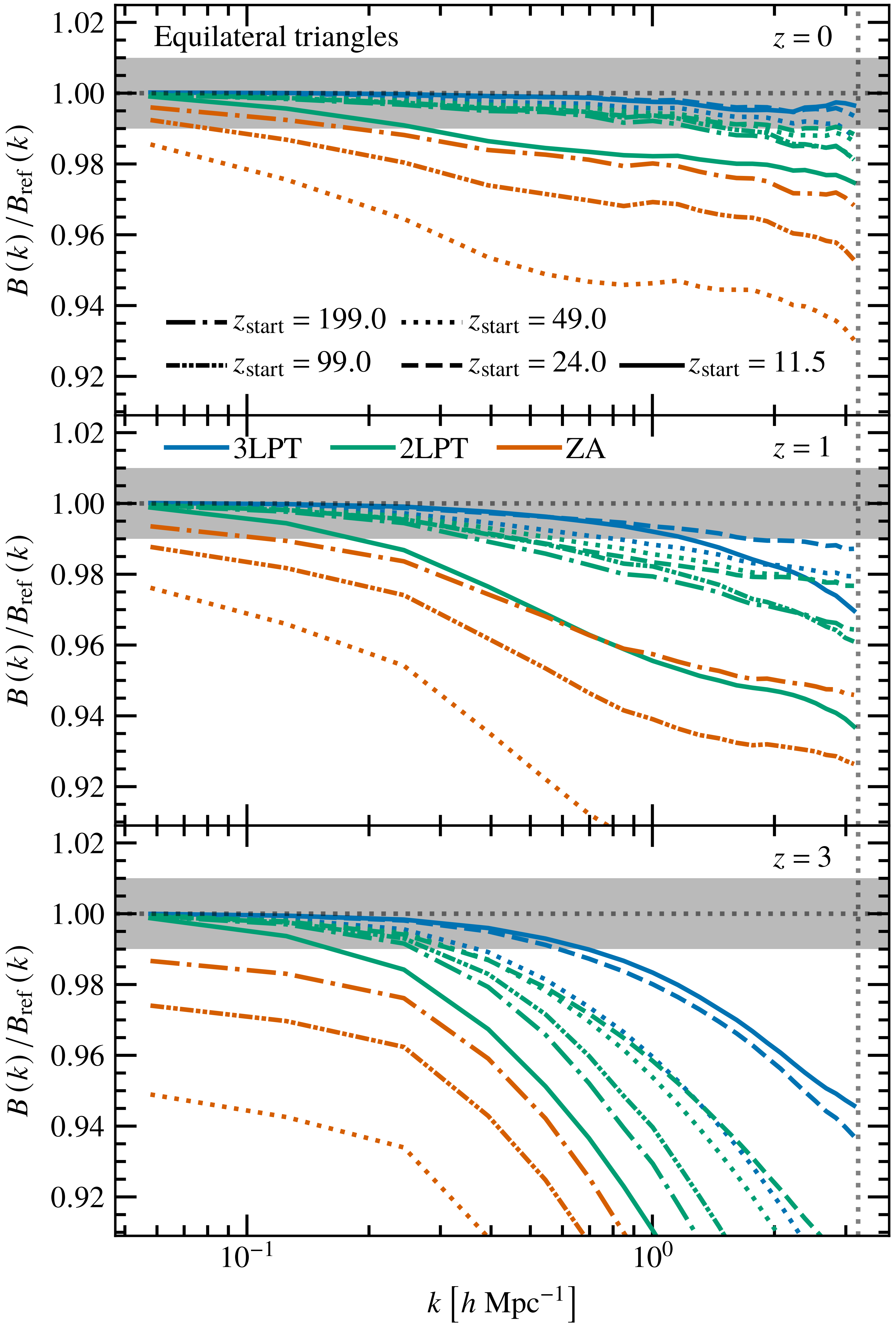
(ref = 3LPT,zstart=24)

Impact of nLPT vs. discreteness on low-z spectra

power spectrum



bispectrum



discreteness always dominates when starting@z too high (cf also Garrison+2016)

best results with high order LPT and low starting redshift (counter to common lore!)

Michaux+2020

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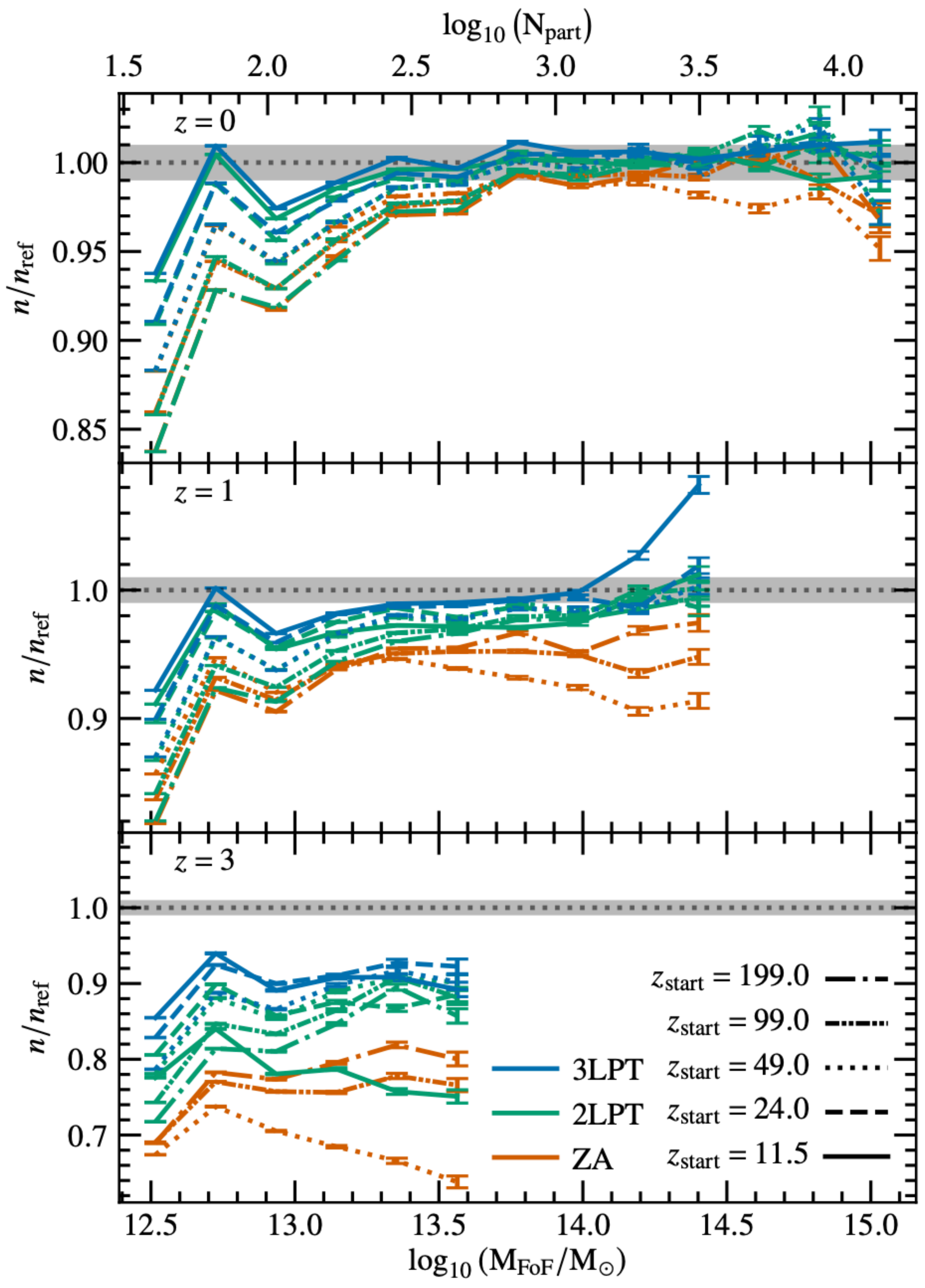
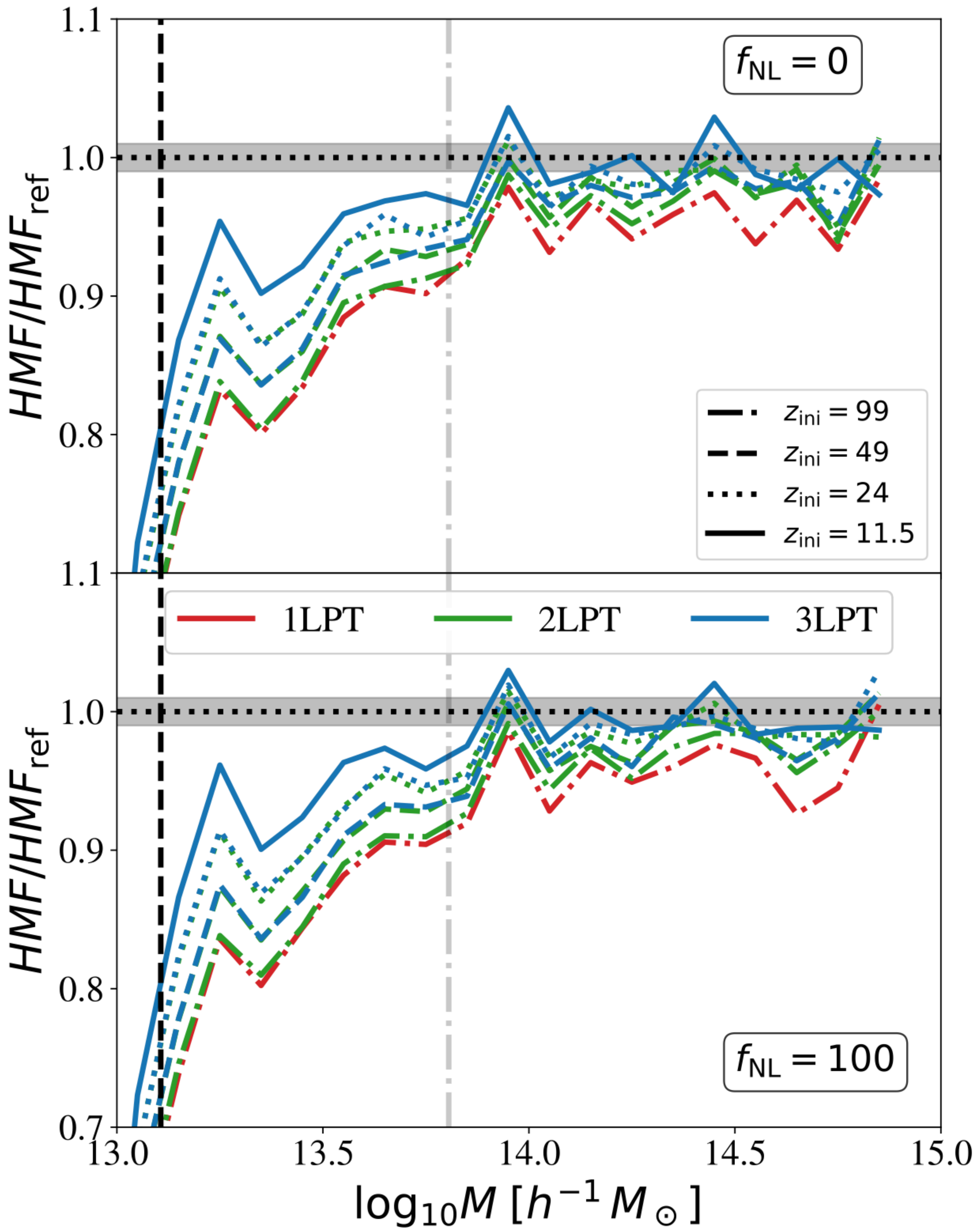
Impact of nLPT on mass functions



Michaux+2020



Adame+2025

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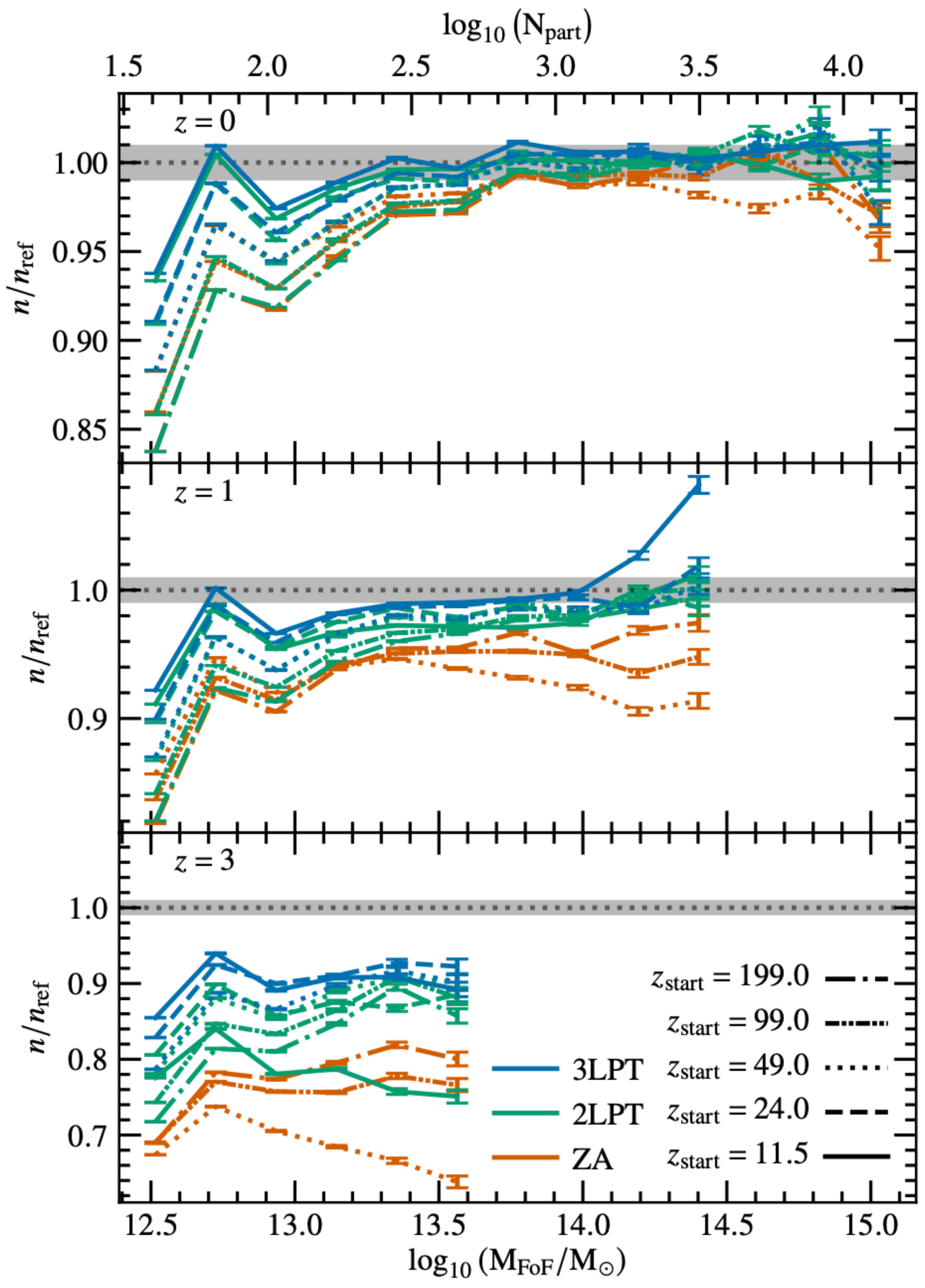

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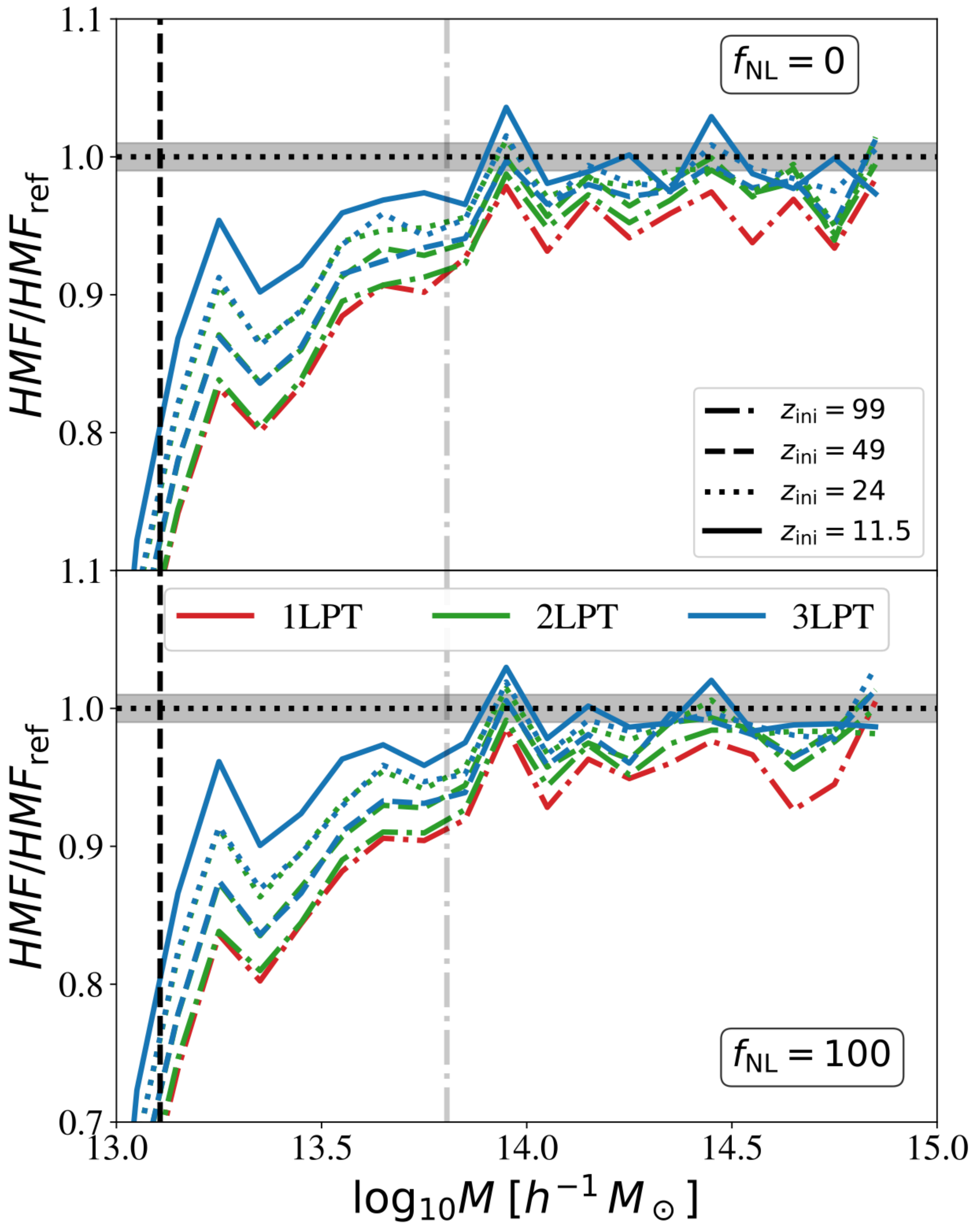

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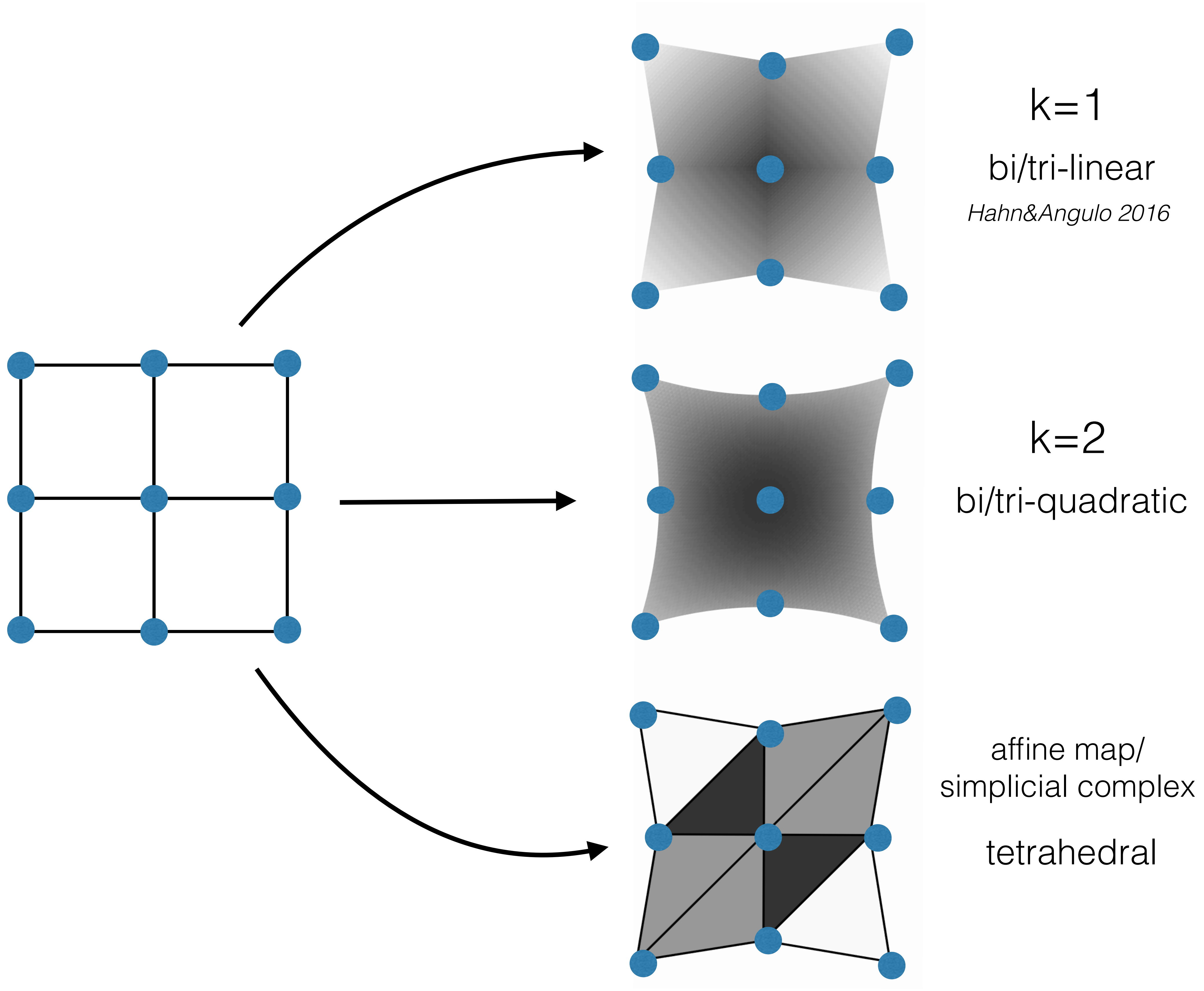
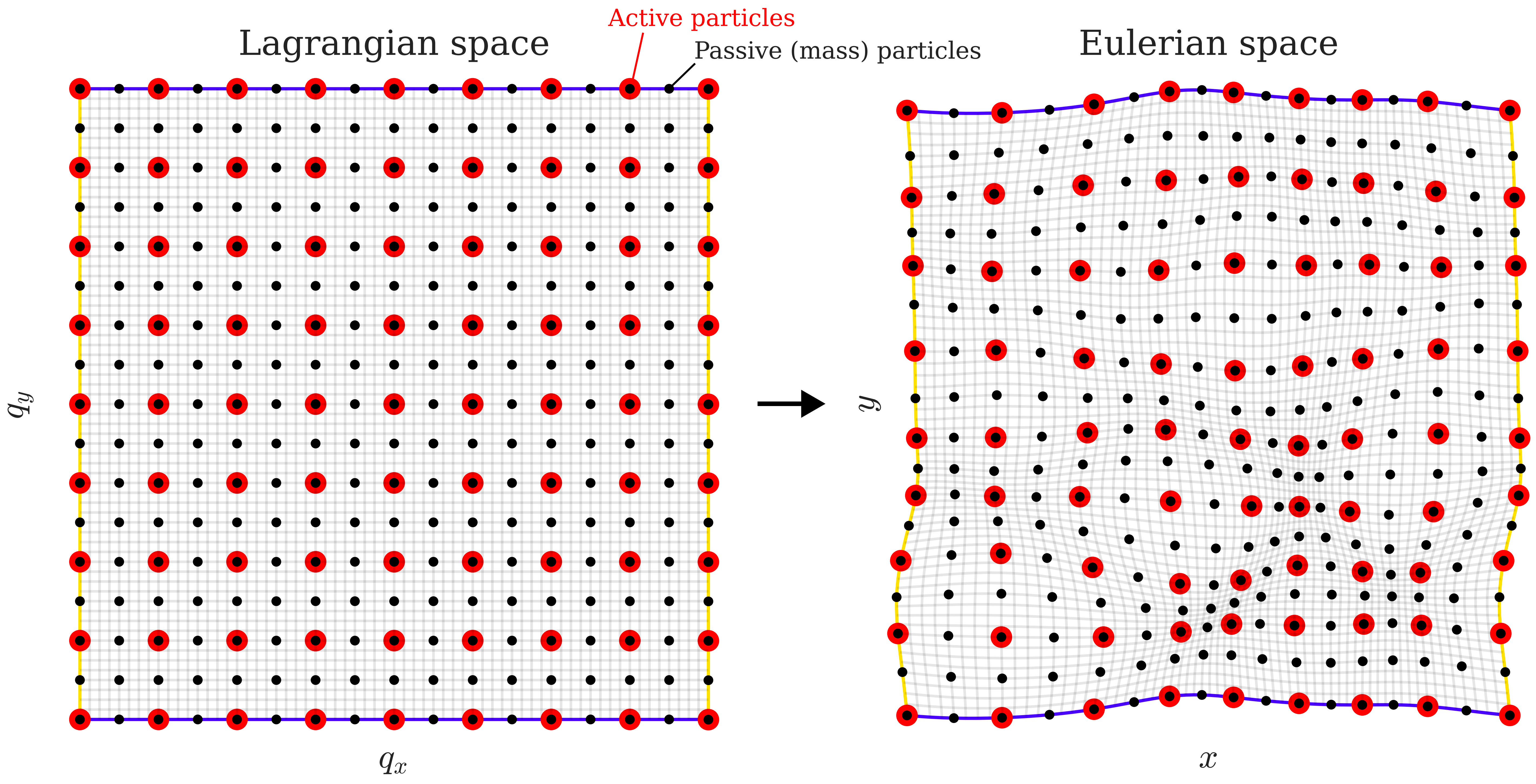
(ref = 3LPT,zstart=24)

Control discreteness errors by sheet interpolation

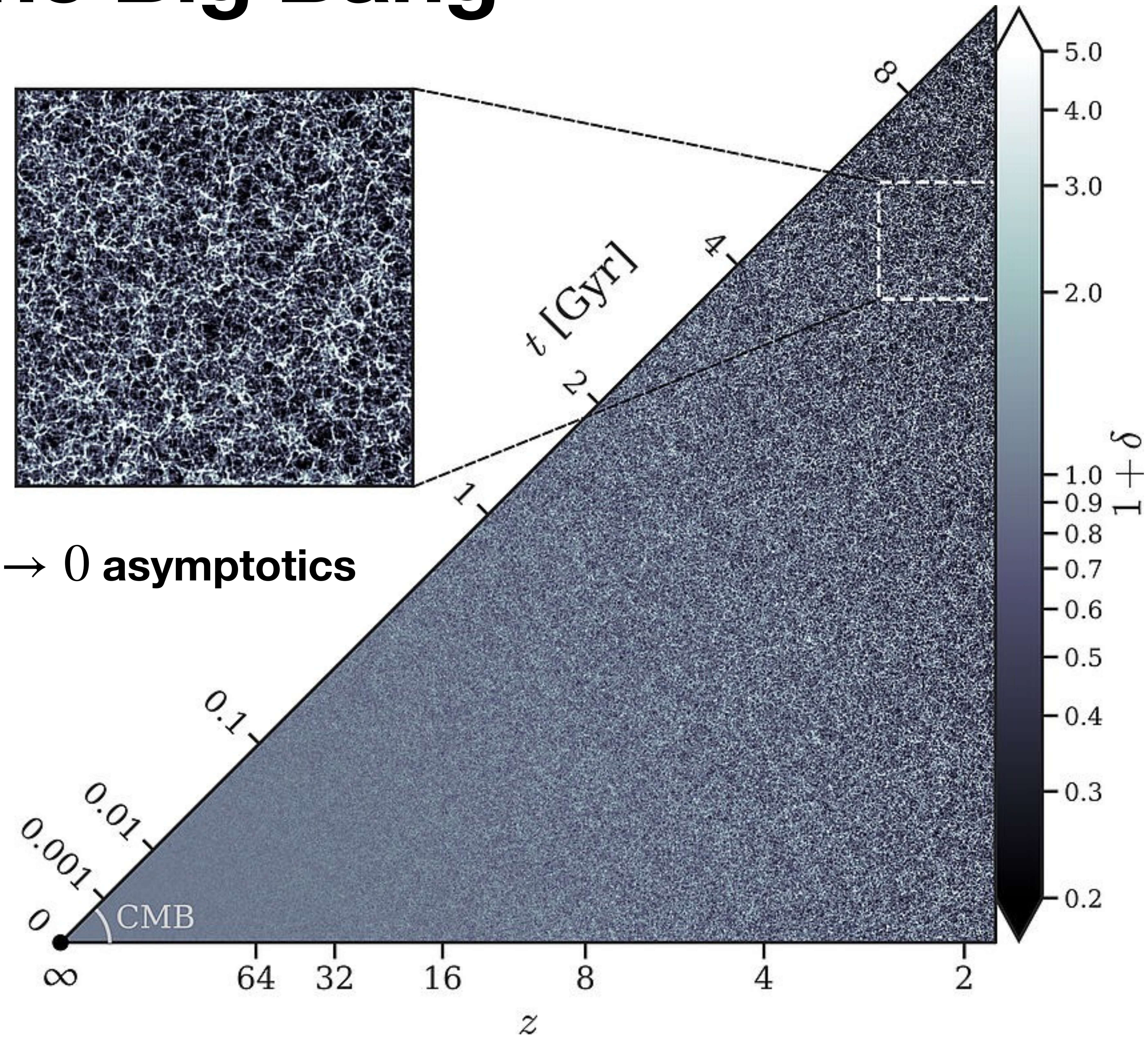




or spectral interpolation (Stücker+2020; List, OH, Rampf 2024)

element-wise interpolation: Abel, Hahn & Kaehler 2012, Schandarin et al. 2012, Sousbie&Colombi 2015, Hahn&Angulo 2016

“Simulations from the Big Bang”



Unlike standard N-body leapfrogs,
LPT-informed integrators have correct $a \to 0$ asymptotics
(no momentum blowup)

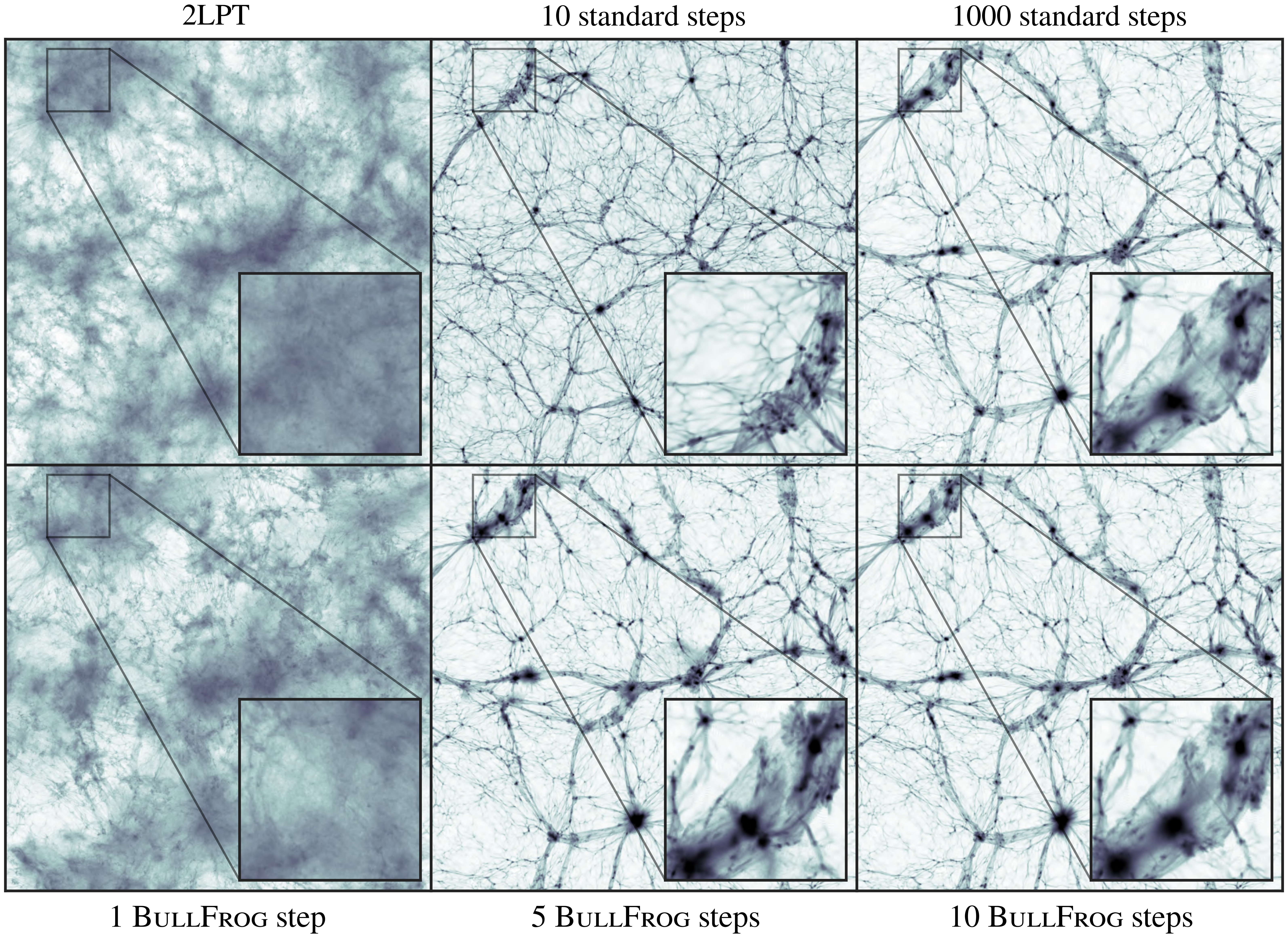
List, OH, Rampf, PRL, 2024
2409.19049

Perturbation theory informed integrators

black board derivation

BullFrog integrator as UV complete multi-step LPT

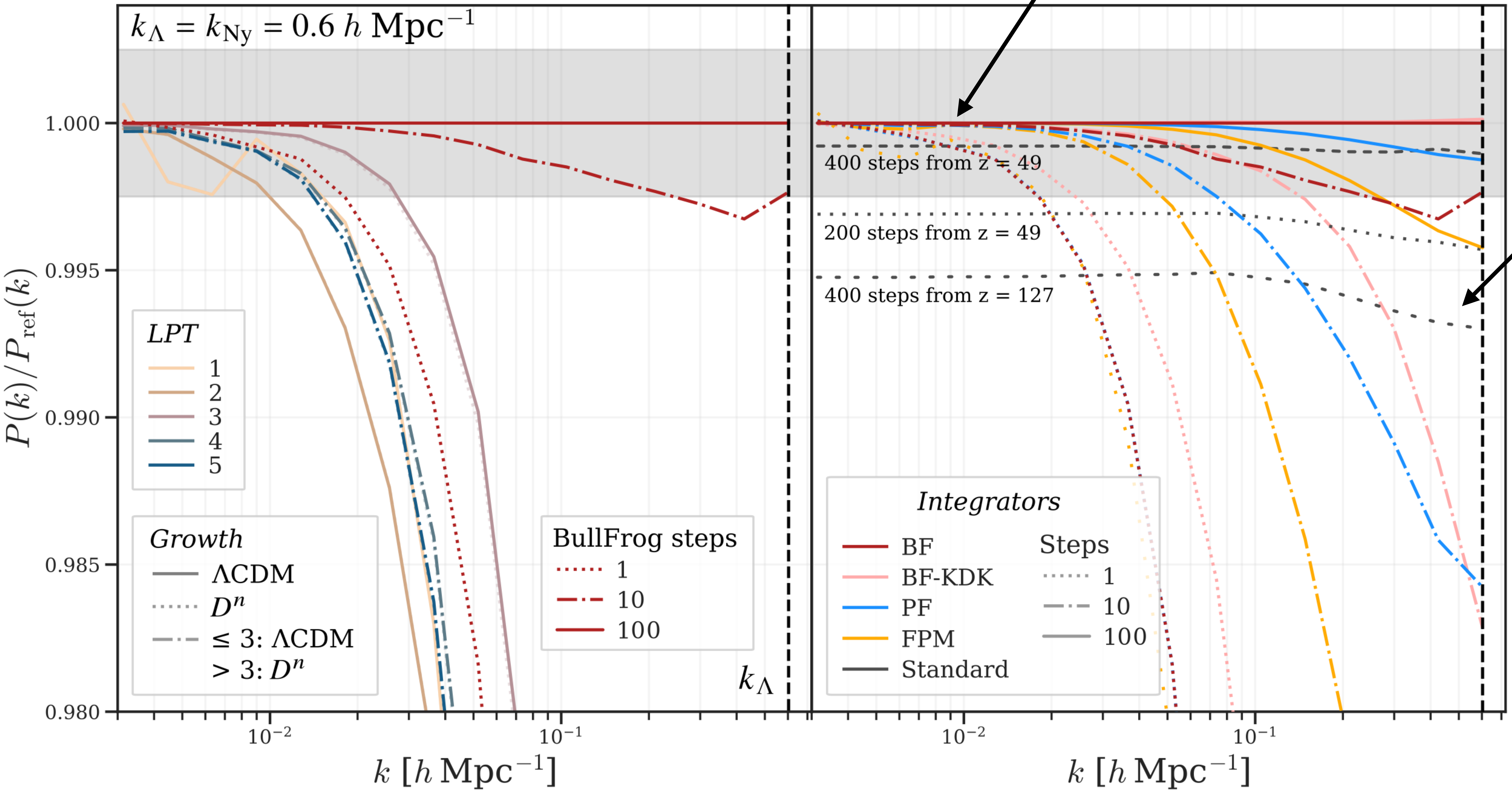
few time steps
=
PT regime (but UV complete)



many time steps
=
just an N-body simulation

Rampf, List, OH, 2025

BullFrog🐸 integrator



"new n-body"
low-k limit always
accurate

"old n-body"
convergence speed
independent of k

Rampf, List, OH, 2025

ICs for two fluid systems: CDM + baryons

MUSIC ecosystem

zoom ICs in the cloud:

https://cosmicweb.eu/

ZOOM IC generator

https://github.com/cosmo-sims/MUSIC2

high-precision full volume IC generator

https://github.com/cosmo-sims/monofonIC

(fast version incl. bullfrog on request)

monofonIC demonstration