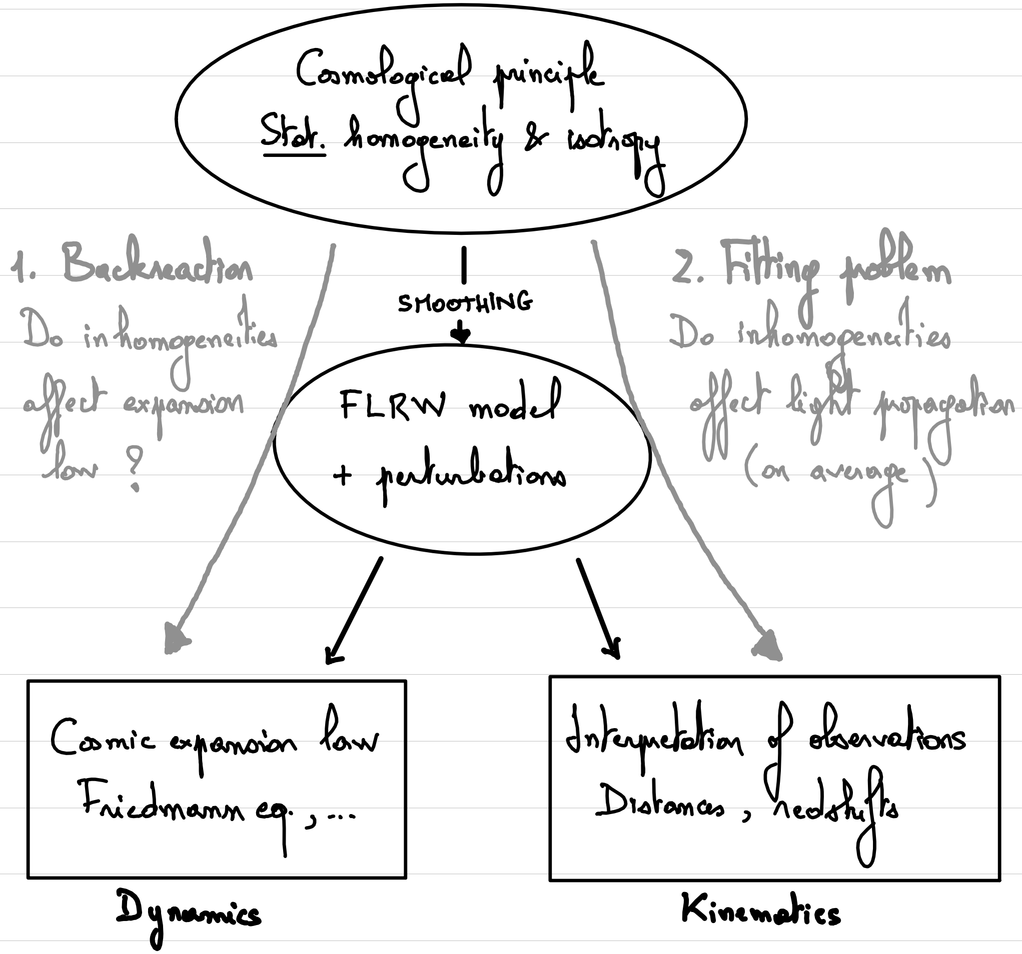
a tiny little bit of

COSMOLOGY BEYOND FLRW

Motivation (5')



1. Backreaction problem

a) General idea (5') total 10'

- FLRW supposes smoothing (Ellis, 1984)
- GR is non-linear

$$\langle E_{\mu\nu}[g_{\mu\nu}]\rangle = 8\pi G \langle T_{\mu\nu}\rangle \underset{\uparrow}{\neq} E_{\mu\nu}[\langle g_{\mu\nu}\rangle]$$

what we do with FLRW

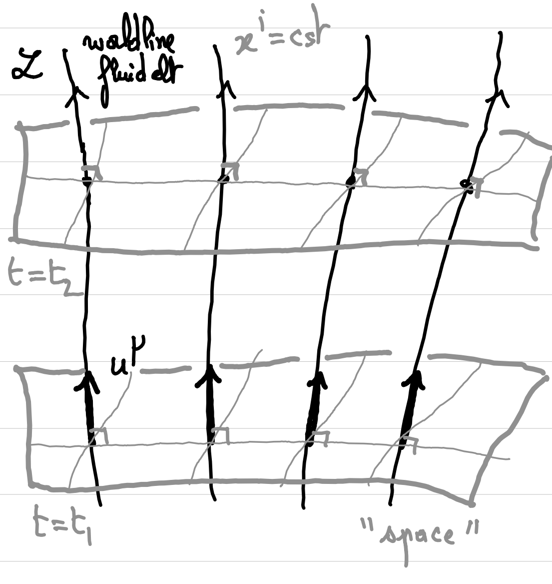
- Coincidence : cosmic acceleration occurs in late univ., when it is most inhomogenous.

↳ backreaction of inhomogeneities ?

b) Buchert's formalism (20') total 30'

- Description of inhom. flow
  - dust ($p=0$)
  - four-velocity field u^μ
  - free fall $u^\nu \nabla_\nu u^\mu = 0$
  - irrotational $\nabla_\mu u_\nu = \nabla_\nu u_\mu$

    ↳ hypersurface orthogonal

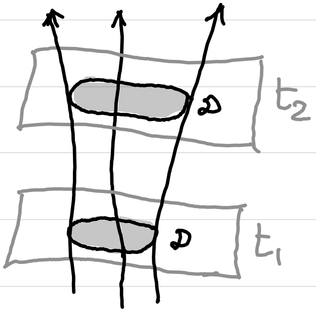


- **Comoving - synchronous coordinates**

$$\begin{cases} t : & \text{proper time along } \mathcal{L} \\ x^i : & \text{labels for } \mathcal{L} \end{cases}$$

$$ds^2 = -dt^2 + h_{ij}(t, \enclose{circle}{x^k})\, dx^i dx^j$$

- **Volume of comoving region $\mathcal{D}$**



$$V_\mathcal{D}(t) = \int_\mathcal{D} d^3x \sqrt{h} \qquad h \equiv \det[h_{ij}]$$

- **Raychaudhuri equation**

local exp. rate $\underset{\sim 3H}{\theta} \equiv \frac{1}{\delta V}\frac{d\delta V}{dt} = \frac{1}{\sqrt{h}}\frac{d\sqrt{h}}{dt} = \frac{1}{2} h^{ij} h_{ij,t}$

Ⓔ $\theta = \nabla_\mu u^\mu = \nabla_i u^i$

more generally: $\nabla_i u_j = \frac{1}{3}\theta h_{ij} + \sigma_{ij} \qquad 0 = h^{ij}\sigma_{ij}$

$\circ \to \bigcirc \qquad \bigcirc \to 0$

$$\begin{aligned} \frac{d\theta}{dt} &= u^\nu \nabla_\nu \nabla_\mu u^\mu \\ &= u^\nu \nabla_\mu \nabla_\nu u^\mu + u^\nu R^\mu{}_{\sigma\nu\mu} u^\sigma \\ &= \nabla_\mu(\cancel{u^\nu \nabla_\nu u^\mu}) - \nabla_\mu u_\nu \nabla^\mu u^\nu - R_{\mu\nu} u^\mu u^\nu \end{aligned}$$

Ⓔ

$$\boxed{\frac{d\theta}{dt} = -\frac{1}{3}\theta^2 - 2\sigma^2 - 4\pi G\rho + \Lambda} \qquad 2\sigma^2 \equiv \sigma^{ij}\sigma_{ij}$$

Raychaudhuri.

• Global expansion law : backreaction

Effective scale factor $a_D(t) \propto V_D^{1/3}(t)$

(E) use volume-averaged Raychaudhuri and algebra :

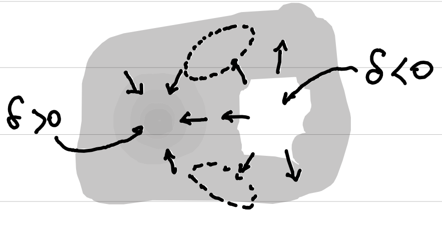
$$\boxed{\begin{aligned} \frac{\ddot{a}_D}{a_D} &= -\frac{4\pi G}{3}\langle\rho\rangle_D + \frac{\Lambda}{3} + \enclose{circle}{\mathcal{B}} \\ \mathcal{B} &= \frac{1}{3}\left[\frac{2}{3}\left(\langle\theta^2\rangle_D - \langle\theta\rangle_D^2\right) - 2\langle\sigma^2\rangle_D\right] \end{aligned}}$$

Buchert's equation

$$\langle X\rangle_D = \frac{1}{V_D}\int_D d^3x\,\sqrt{h}\,X\,.$$

• Caveats

- formalism only for scalars → missing info → $\mathcal{B}$ not quantified
- $\langle\theta^2\rangle_D - \langle\theta\rangle_D^2$ and $\langle\sigma^2\rangle_D$ tend to compensate



In Newtonian physics, they exactly compensate

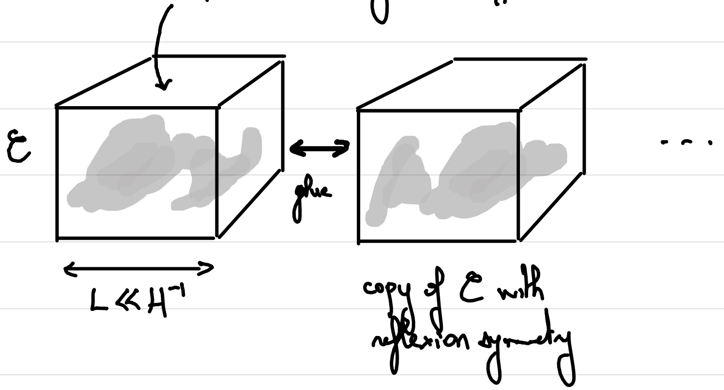
$$\mathcal{B} = \frac{1}{V_D}\int_{\partial D}\cdots$$

↳ Backreaction must be relativistic

c) Relativistic backreaction (10') total 40'

- Post-Newtonian patchwork universe (Clifton & Sanghai '16)

PN expansion : $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon^2 \gamma_{\mu\nu}$ $\varepsilon \sim v/c$



Glue copies of ε ⟶ patchwork universe
Global expansion law emerges from small-scale physics

$$\frac{\ddot{L}}{L} = -\frac{4\pi G}{3}\langle\rho\rangle_\varepsilon + \frac{\Lambda}{3} + \mathcal{B}$$

$$\boxed{\mathcal{B} \simeq \underset{\text{repulsive}}{+} \frac{2}{3}\frac{(\Omega_m H_0)^2}{\underset{\text{behaves like radiation}}{a^4}} \underset{\text{suppressed}}{(H_0 L_0)^2} = \mathcal{O}(\varepsilon^4)}$$

↳ Negligible backreaction

- **Effective field theory of GR** (Green & Wald 2011, 2014)

$$g_{\mu\nu} = \underset{\substack{\nearrow \\ \text{Large scales}}}{\bar{g}_{\mu\nu}} + \underset{\substack{\nwarrow \\ \text{Small scales}}}{\gamma_{\mu\nu}}$$

$g_{\mu\nu}$ satisfies EFE

$\gamma_{\mu\nu} \ll 1$

$\nabla_\rho \gamma_{\mu\nu}$ finite

$\nabla_\rho \nabla_\sigma \gamma_{\mu\nu}$ arbitrary

$$E_{\mu\nu}[\bar{g}_{\mu\nu}] = 8\pi G (\bar{T}_{\mu\nu} + t_{\mu\nu})$$

↳ backreaction from $\gamma_{\mu\nu}$

$t^\mu_\mu = 0$ → integrated-out GWs.

- **Gevolution** (Adamek et al. 2016) : relativistic N-body simulation without pre-determined backg. ev.
  ↳ Def. of $a(t)$ updated at each time step.
  Backreaction $< 10^{-4}$.

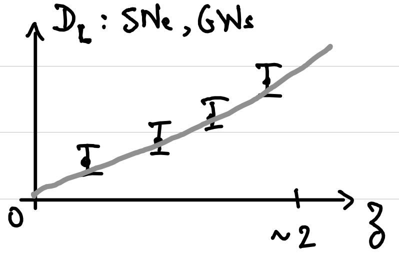
40' at this point

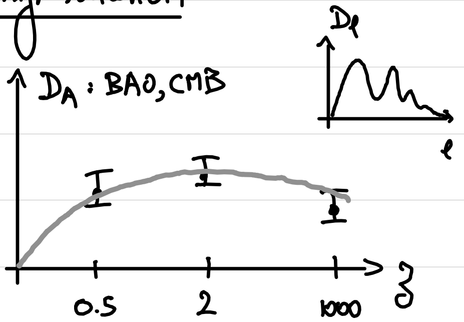
2. The fitting problem

measurements

~~Physical distances, velocities~~ $\xleftarrow[\text{theoretical model}]{}$ redshift, angle, brightness

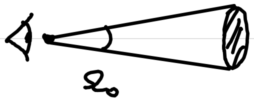
Archetypal case : distance - redshift relation



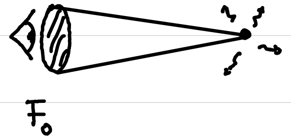


a) Distance - redshift relation (10') total 50'

- Defs $1+z = \frac{\omega_s}{\omega_o}$ (Light rays)



$A_s = D_A^2\,\Omega_o$ (Light beams)



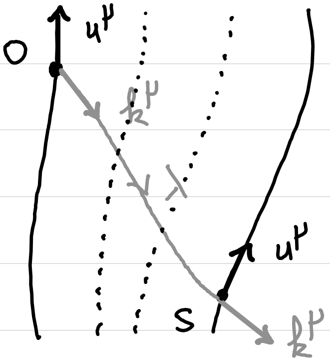
$L_s = F_o\, 4\pi D_L^2$

$D_L = (1+z)^2 D_A$ (always!)

in FLRW

$$\bar{D}_A(z) = \frac{1}{1+z}\,\frac{\sinh\sqrt{-K}\,\chi(z)}{\sqrt{-K}} \qquad \chi(z) = \int_0^z \frac{dz'}{H(z')}$$

- In general: two relations $z(\lambda)$ and $D_A(\lambda)$

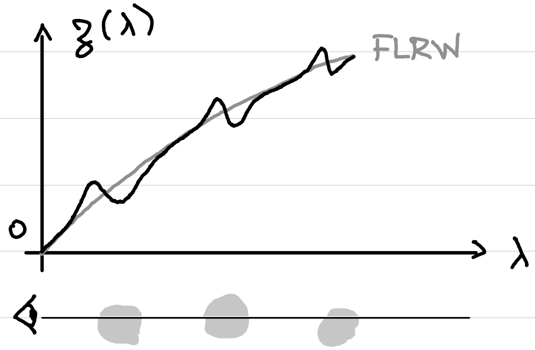


Light ray : $k^\nu \nabla_\nu k^\mu = \frac{Dk^\mu}{d\lambda} = 0$

$$1+z = \frac{(u_\mu k^\mu)_s}{(u_\mu k^\mu)_o}$$

Ⓔ $$\frac{dz}{d\lambda} = k^i k^j \nabla_i u_j = (1+z)^2 \underset{\uparrow}{H_{\parallel}}$$

expansion rate along $\hat{k}$

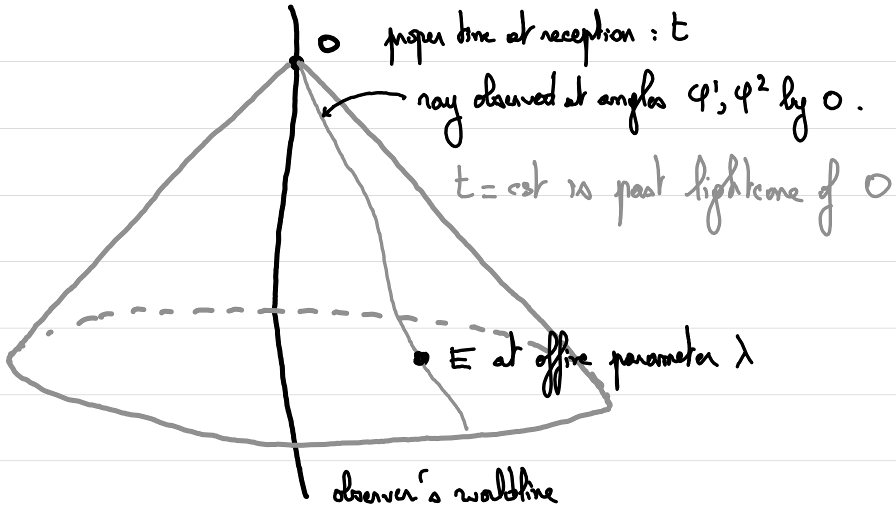


If no backreaction, just wiggle around FLRW.

What about $D_A(\lambda)$? Need to follow $A(\lambda)$, so study light beams.

b) Light beams in GR (15') total 65'

- Observational coordinates $E(t, \lambda, \varphi^a)$



- Metric in observational

$$ds^2 = -N\,dt^2 - dt\,d\lambda + h_a\,dt\,d\varphi^a + \boxed{S_{ab}\,d\varphi^a d\varphi^b}$$

metric on the lightcone $t = cst$

- Cross-sectional area of light beam

$$A(\lambda) = \int_{\mathcal{B}} d^2\Omega\, \underset{\uparrow}{\sqrt{S}} \qquad S \equiv \det[S_{ab}]$$

recognise D_A^2

• Optical Raychaudhuri equation

$$\underset{\substack{\uparrow \\ \text{wavefront expansion rate}}}{\Theta} \equiv \frac{1}{\delta A}\frac{d\delta A}{d\lambda} = \frac{1}{\sqrt{S}}\frac{d\sqrt{S}}{d\lambda} = \frac{1}{2} S^{ab} S_{ab,\lambda}$$

Ⓔ $\Theta = \nabla_\mu k^\mu = \nabla_a k^a$

More generally, $\nabla_a k_b = \frac{1}{2}\underset{\substack{\uparrow \\ \text{expansion rate of beam}}}{\Theta} S_{ab} + \underset{\substack{\nwarrow \\ \text{shear rate} \\ \sim d\gamma/d\lambda}}{\Sigma_{ab}}$

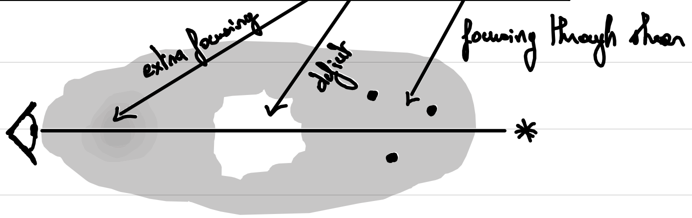
Ⓔ $$\frac{d\Theta}{d\lambda} = -\frac{1}{2}\Theta^2 - 2\Sigma^2 - \underbrace{R_{\mu\nu}k^\mu k^\nu}_{8\pi G\rho(1+z)^2} \qquad 2\Sigma^2 \equiv \Sigma^{ab}\Sigma_{ab}$$

optical Raychaudhuri eq.

Substitute $\Theta = \frac{2}{D_A}\frac{dD_A}{d\lambda}$, get

$$\boxed{\frac{1}{D_A}\frac{d^2 D_A}{d\lambda^2} = -4\pi G(1+z)^2\rho - \Sigma^2}$$

Focusing theorem

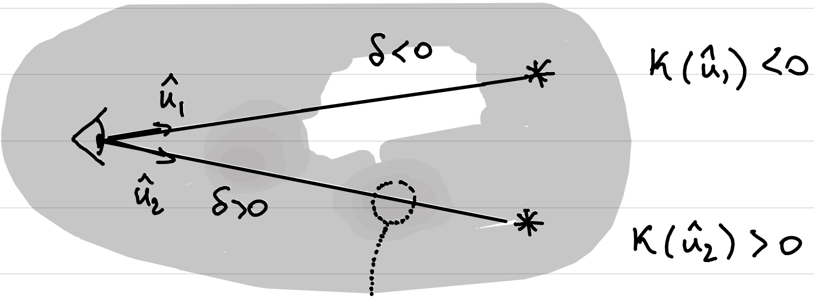


Rk: No effect from Λ.

c) Optics in a lumpy Universe (10') total 75'

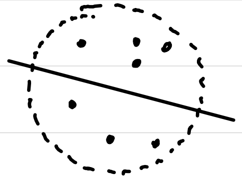
- **Perturbation theory** $ds^2 = -(1+2\Phi)dt^2 + a^2(1-2\Phi)d\vec{x}^2$

$$\frac{\delta D_A}{D_A} = -4\pi G\,\bar{\rho}_0 \int_0^{\chi_s} d\chi\, \frac{\chi(\chi_s-\chi)}{\chi_s}\,\frac{\delta(\eta_0-\chi, \chi, \hat{u})}{a(\eta_0-\chi)} \equiv -K(z_s, \hat{u})$$



Compensate on average $\langle K\rangle \simeq 0$.

- **Zeldovich's empty beam**

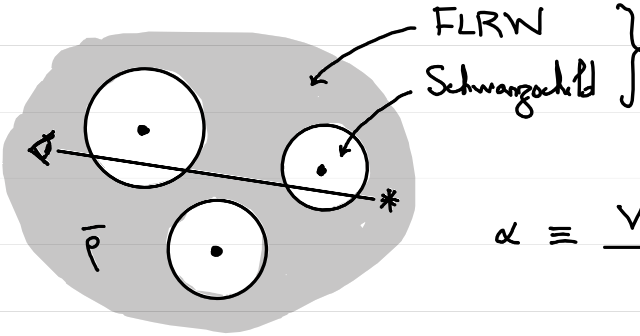


Beam passes | through vacuum $\rho \simeq 0$
| far from matter lumps $\Sigma^2 \simeq 0$

$$\frac{d^2 D_A}{d\lambda^2} \simeq 0 \Rightarrow D_A^{EB} = \lambda = \int_0^z \frac{dz'}{(1+z')H(z')} < \bar{D}_A$$

In Zeldovich's model, sources at z appear consistently smaller and dimmer than in FLRW.

- Swiss-cheese models



Can be joined if mass conserved

$$\alpha \equiv \frac{V_{FLRW}}{V}$$

For opaque lumps, $D_A(z)$ very well described by

$$\frac{1}{D_A}\frac{d^2 D_A}{d\lambda^2} \simeq -4\pi G(1+z)^2\alpha\bar{\rho} - \cancel{\Sigma^2}$$

neglected

Kantowski - Dyer - Roeder approximation

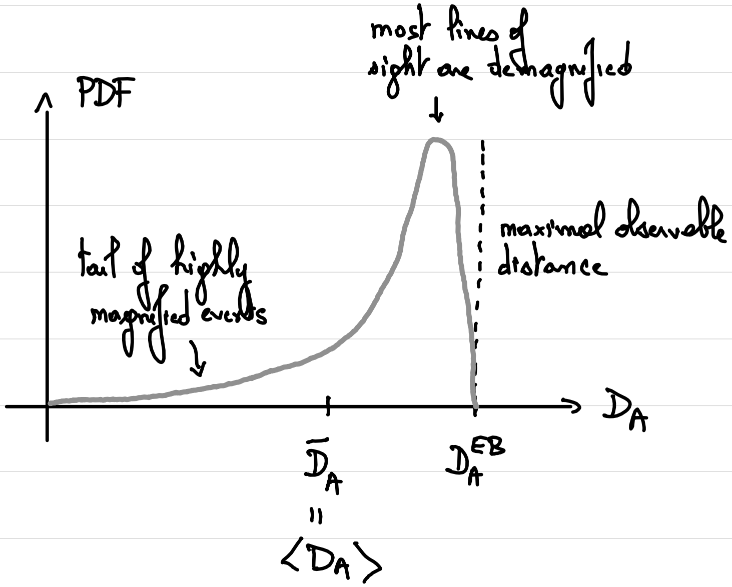
- What about shear ?

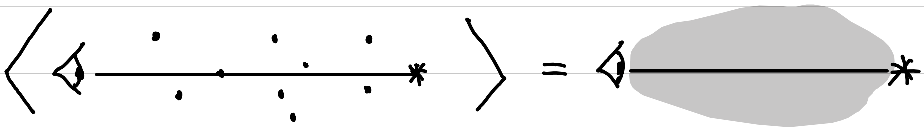
$$\left(\frac{\delta D_A}{D_A}\right)_{\text{due to } \Sigma^2}(z=1) \simeq -\frac{2GM}{H_0 R^2} \times 10^{-3}$$

galax. ~ 1

stars $\sim 10^{10}$

So the effect of shear can be very large.
Weinberg (1976) : Σ^2 compensates exactly the lack of matter within lumps on average.





<u>Rk</u>: Weinberg's calculation done at first order in density.

d) Observational averages (10') total 85'

- <u>Magnification theorems</u>

observed size of source

$$\mu \equiv \frac{d^2\Omega_o}{d^2\Omega_s} = \left(\frac{\bar{D}_A}{D_A}\right)^2$$

size it would have in FLRW

$$\langle \mu \rangle_s = \frac{1}{4\pi}\int_{\mathcal{S}^2} d^2\Omega_s \, \mu = \frac{1}{4\pi}\int_{\mathcal{S}^2} d^2\Omega_o = 1 .$$

$$\langle \mu^{-1} \rangle_o = \frac{1}{4\pi}\int_{\mathcal{S}^2} d^2\Omega_o \, \mu^{-1} = \frac{1}{4\pi}\int_{\mathcal{S}^2} d^2\Omega_s = 1 .$$

Magnification theorems

Ⓔ Remains true with multiple imaging (e.g. Breton & Fleury 21)

- Bias on distance

$$Ⓔ \quad \langle D_A \rangle_s = \Big(1 + \underbrace{\frac{3}{2}\langle \kappa^2 \rangle}_{} + \dots \Big) \bar{D}_A$$

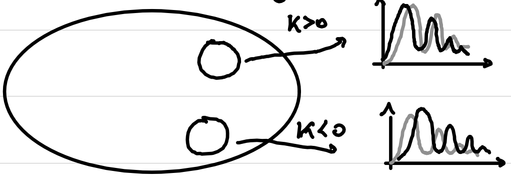
$\simeq 10^{-3}$ at $z = 2$. 5 % at $z = 10^3$

- The distance-to-the-CMB controversy

Is there ~ 1% error on CMB analysis ? [Clarkson + 2014]

3 issues with that conclusion :

(i) Not the right observable $\longrightarrow \theta_* = \dfrac{r_s}{D_A(z_*)}$

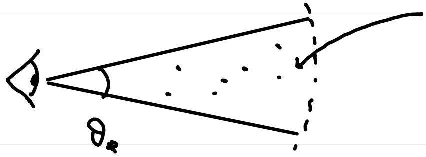


$$\langle \theta_* \rangle = \bar{\theta}_* \langle \sqrt{\mu} \rangle_s$$
$$= \bar{\theta}_* \left(1 - \frac{1}{4}\langle \kappa^2 \rangle + \dots \right)$$

(ii) Small-scale perturbations do not contribute

$$\langle \kappa^2 \rangle = \frac{1}{4\pi} \sum_{\ell=0}^{\infty} (2\ell+1)\, C_\ell^\kappa$$

$$C_\ell^\kappa = (4\pi G \bar{\rho}_0)^2 \int_0^{\chi_*} d\chi \left[\frac{\chi(\chi_*-\chi)}{a(\eta_0-\chi)\chi_*} \right]^2 P_m\left(\chi, \frac{\ell+1/2}{\chi}\right)$$



fluctuations on scales $\ell > \pi/\theta_*$ do not contribute !

$$\langle \kappa^2 \rangle_{\ell_{max}\to\infty} = 3\% \longrightarrow \langle \kappa^2 \rangle_{\ell_{max}=\frac{\pi}{\theta_*}} = 7\times 10^{-4}.$$

(iii) The effect of lensing is already taken into account in current CMB analyses.

85' at that point

Conclusion (5')

- Backreaction : effect of inhomogeneities on expansion law
  ↳ small in GR
  ⚠ Not in alternative theories, especially if screening

- Fitting problem : effect of inhomogeneities on observations
  ↳ small effect on average
  ⚠ Significant on individual obs., small datasets.

Testing the cosmological principle : intriguing observations.

- Dipole tension (Secrest et al. 2021)
- Regions in CMB (Fosalba & Gaztañaga 2020)

——— end.