Second Order Cosmological Perturbation Theory over the Geodesic Light-Cone Background

Pierre Béchaz
[based on PB, Giuseppe Fanizza, Giovanni Marozzi and Matheus Medeiros, arXiv:2508.XXXXX]

University of Pisa, Department of Physics "Enrico Fermi" and INFN, Section of Pisa

Les Houches Summer School "The Dark Universe"

UNIVERSITÀ DI PISA

INFN PISA

Plan for the talk

Building blocks

1. The Geodesic Light-Cone coordinates
2. Perturbation theory on the Light-Cone up to second order

Fixing the GLC gauge

3. Cosmological observables in the GLC gauge
4. The GLC gauge in perturbation theory

Application

5. The observational angular distance-redshift relation

The Geodesic Light-Cone coordinates

★ GLC coordinates [Gasperini, Marozzi, Nugier, Veneziano, JCAP, 1107 (2011) 008]

physically motivated coordinates

$$x^\mu = (\tau, w, \tilde{\theta}^a)\,, \quad a = 1,2$$

$\tau = \text{const.} \leftrightarrow \text{geodesic obs.}$

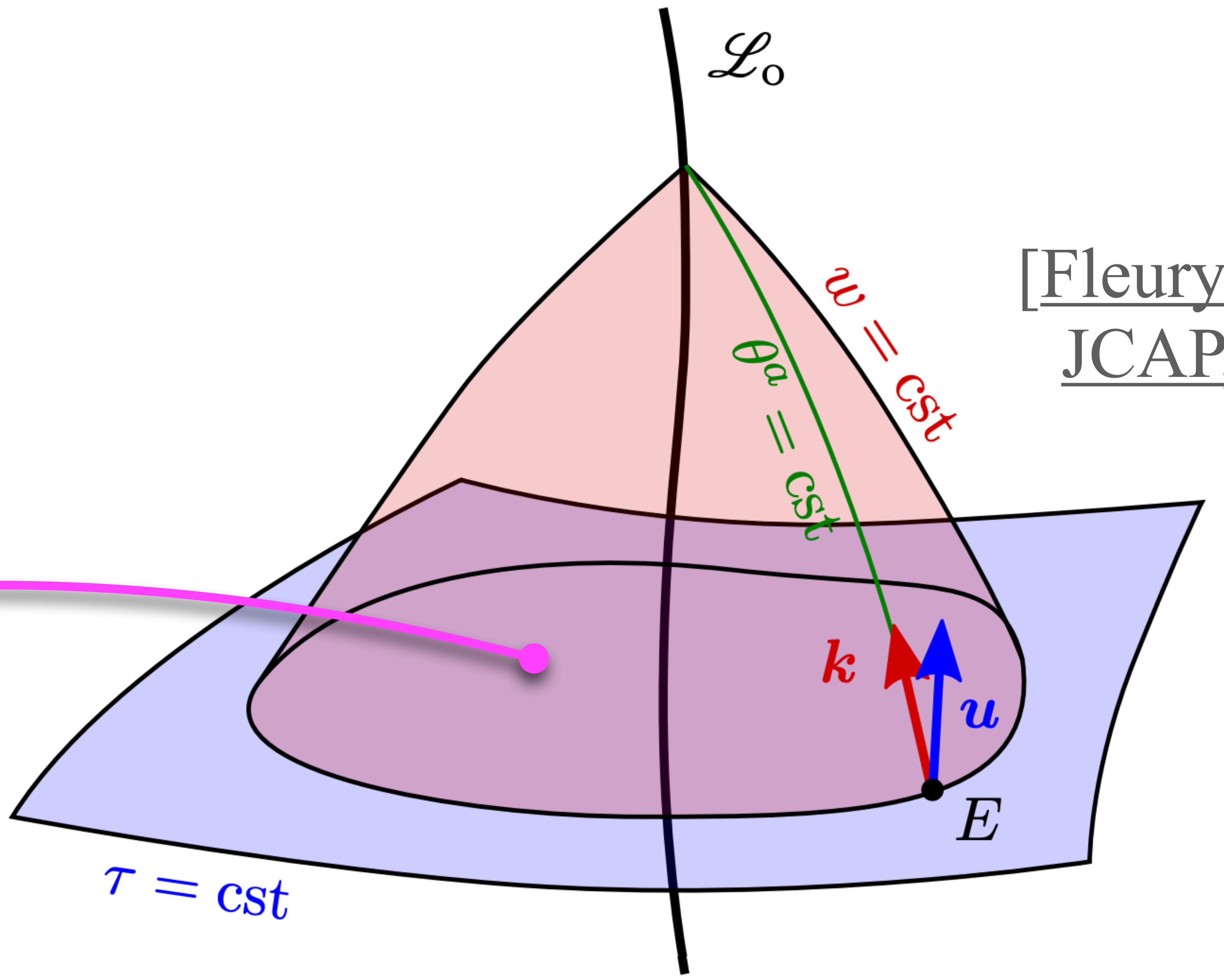
$w = \text{const.} \leftrightarrow \text{past LC}$

angular directions in the sky

★ GLC gauge:

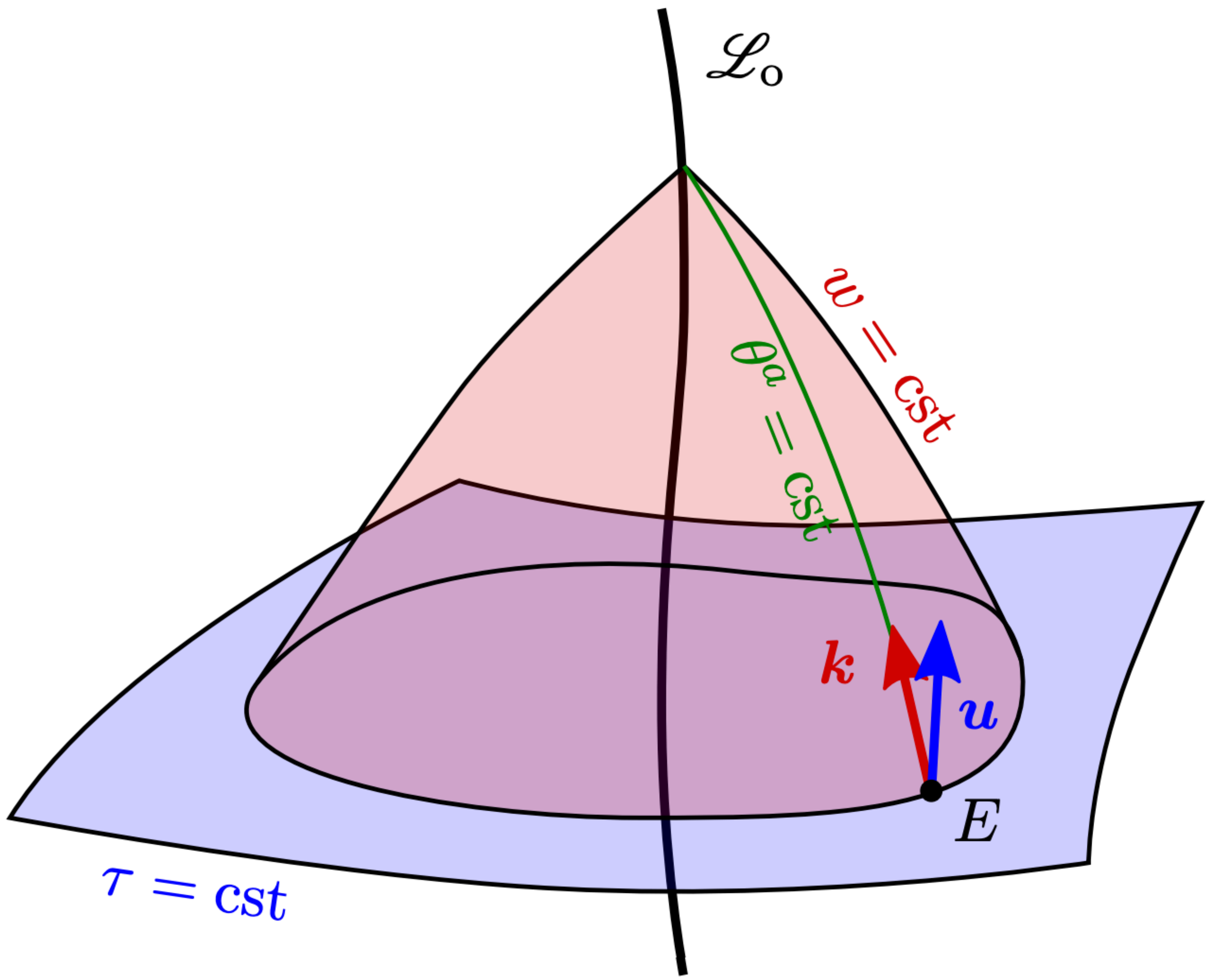
$$\mathrm{d}s^2 = -2\Upsilon \mathrm{d}\tau \mathrm{d}w + \Upsilon^2 \mathrm{d}w^2 + \\ + \gamma_{ab}(\mathrm{d}\tilde{\theta}^a - \mathcal{U}^a \mathrm{d}w)(\mathrm{d}\tilde{\theta}^b - \mathcal{U}^b \mathrm{d}w)$$

induced metric on $\mathbb{S}^2 \ni \tilde{\theta}^a$



[Fleury, Fanizza, Nugier, JCAP, 06 (2016) 008]

Physical meaning



- ⋆ The 4-velocity of a geodesic observer is

$$u^\mu = -\partial^\mu \tau \qquad u^\mu u_\mu = -1$$

proper time

- ⋆ The wave-vector of an incoming photon is

$$k^\mu \propto \partial^\mu w \qquad \partial^\mu w \partial_\mu w = 0$$

- ⋆ Light-like geodesics are *exactly* solved by

$$\tilde{\theta}^a = \mathrm{const.}$$

Standard Perturbation Theory in a nutshell

★ Perturbed FLRW metric:

$$\mathrm{d}s^2 = a^2(\eta)\Big[-\left(1+2(\phi^{(1)}+\phi^{(2)})\right)\mathrm{d}\eta^2 - 2(\mathcal{B}_r^{(1)}+\mathcal{B}_r^{(2)})\mathrm{d}\eta\mathrm{d}r - 2(\mathcal{B}_a^{(1)}+\mathcal{B}_a^{(2)})\mathrm{d}\eta\mathrm{d}\theta^a + $$
$$+ (1+\mathcal{C}_{rr}^{(1)}+\mathcal{C}_{rr}^{(2)})\mathrm{d}r^2 + 2(\mathcal{C}_{ra}^{(1)}+\mathcal{C}_{ra}^{(2)})\mathrm{d}r\mathrm{d}\theta^a + (\bar{\gamma}_{ab}^{\mathrm{FLRW}}+\mathcal{C}_{ab}^{(1)}+\mathcal{C}_{ab}^{(2)})\mathrm{d}\theta^a\mathrm{d}\theta^b\Big]$$

★ Scalar-Vector-Tensor decomposition:

$$\mathcal{B}_i^{(n)} = \partial_i B^{(n)} + B_i^{(n)}\,, \qquad \nabla^i B_i^{(n)} = 0\,,$$
$$\mathcal{C}_{rr}^{(n)} = -2\psi^{(n)} + 2D_{rr}E^{(n)} + 2\nabla_r F_r^{(n)} + 2h_{rr}^{(n)}\,, \qquad \nabla^i F_i^{(n)} = 0\,,$$
$$\mathcal{C}_{ra}^{(n)} = 2D_{ra}E^{(n)} + 2\nabla_{(r}F_{a)}^{(n)} + 2h_{ra}^{(n)}\,, \qquad h_{ii}^{(n)} = 0\,,$$
$$\mathcal{C}_{ab}^{(n)} = -2\psi^{(n)}\,\bar{\gamma}_{ab}^{\mathrm{FLRW}} + 2D_{ab}E^{(n)} + 2\nabla_{(a}F_{b)}^{(n)} + 2h_{ab}^{(n)}\,, \qquad \nabla^i h_{ij}^{(n)} = 0\,.$$

where $D_{ij} \equiv \nabla_{(i}\nabla_{j)} - \frac{1}{3}\bar{\gamma}_{ij}\Delta_3$ and $i = (r,\theta,\phi)$

LC perturbation theory from SPT

GOAL

Build a perturbation theory on the light-cone starting from the FLRW background metric

SPT: $\begin{cases} y^\mu = (\eta, r, \theta^a) \\ g_{\mu\nu} = \bar{g}_{\mu\nu} + g^{(1)}_{\mu\nu} + g^{(2)}_{\mu\nu} \end{cases}$

perturbed LC: $\begin{cases} x^\mu = (\tau, w, \tilde{\theta}^a) \\ f_{\mu\nu} = \bar{f}_{\mu\nu} + f^{(1)}_{\mu\nu} + f^{(2)}_{\mu\nu} \end{cases}$

$$\mathrm{d}\eta = \frac{\mathrm{d}\tau}{a(\tau)}$$

$$r = w - \eta(\tau)$$

$$\theta^a = \tilde{\theta}^a$$

background diffeomorphism

Perturbation theory on the Light-Cone

★ Background LC metric from FLRW one:

$$\bar{f}_{\mu\nu} = \frac{\partial y^\rho}{\partial x^\mu}\frac{\partial y^\sigma}{\partial x^\nu}\,\bar{g}_{\rho\sigma} \longrightarrow \bar{f}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = a^2(\tau)\left[-\frac{2}{a}\mathrm{d}\tau\mathrm{d}w + \mathrm{d}w^2 + \bar{\gamma}_{ab}\mathrm{d}\tilde{\theta}^a\mathrm{d}\tilde{\theta}^b\right]$$

★ Perturbed LC metric:

$$\mathrm{d}s^2 = a^2(\tau)\Big[(L^{(1)} + L^{(2)})\mathrm{d}\tau^2 - \frac{2}{a}\big(1 - a(M^{(1)} + M^{(2)})\big)\mathrm{d}\tau\mathrm{d}w + 2(V_a^{(1)} + V_a^{(2)})\mathrm{d}\tau\mathrm{d}\tilde{\theta}^a +$$

$$+ (1 + N^{(1)} + N^{(2)})\mathrm{d}w^2 + 2(U_a^{(1)} + U_a^{(2)})\mathrm{d}w\mathrm{d}\tilde{\theta}^a + (\bar{\gamma}_{ab} + \gamma_{ab}^{(1)} + \gamma_{ab}^{(2)})\mathrm{d}\tilde{\theta}^a\mathrm{d}\tilde{\theta}^b\Big]$$

$$\bar{\gamma}_{ab} = \big[w - \eta(\tau)\big]^2 \mathrm{diag}(1, \sin^2\tilde{\theta}^1) \equiv \big[w - \eta(\tau)\big]^2 q_{ab}$$

Scalar-PseudoScalar decomposition

- Define the operators [Fanizza, Marozzi, Medeiros, Schiaffino, JCAP, 02 (2021) 014; Mitsou, Fanizza, Grimm, Yoo, Class. Quantum. Grav. 38 (2021) no. 5 055011]

$$D_{ab} \equiv D_{(a}D_{b)} - \frac{1}{2}q_{ab}D^2 \qquad \tilde{D}_{ab} \equiv \tilde{D}_{(a}D_{b)} = D_{(a}\tilde{D}_{b)} \qquad \tilde{D}_a \equiv \epsilon_a^b D_b$$

- Perturbations are decomposed according their transformation properties under SO(2):

$$V_a^{(n)} = r^2\left[D_a v^{(n)} + \tilde{D}_a \hat{v}^{(n)}\right]$$
$$U_a^{(n)} = r^2\left[D_a u^{(n)} + \tilde{D}_a \hat{u}^{(n)}\right]$$
$$\gamma_{ab}^{(n)} = 2r^2\left[q_{ab}\nu^{(n)} + D_{ab}\mu^{(n)} + \tilde{D}_{ab}\hat{\mu}^{(n)}\right]$$

$$D_a\hat{v}^{(n)} = 0 = D_a\hat{u}^{(n)}$$

with scalar and pseudo-scalar variables.

Remarks on SPS decomposition

PROS

- ★ LC coordinates incorporate all the physical information about the various types of perturbations, already decomposed into E- and B-modes [Fanizza, Marozzi, Medeiros, JCAP, 02 (2023) 015].
- ★ All the inhomogeneities and anisotropies are "embedded" in scalar and pseudo-scalar fluctuations.

CONS

- ★ Already at first order, perturbations are "coupled" to each other.

- ★ Perturbative calculations can be quite involved.

Gauge transformations

- Note that

$\boxed{\text{10 d.o.f.s in the perturbed LC metric}} \longrightarrow \boxed{\text{gauge not yet specified}}$

- Consider the infinitesimal coordinate transformation

$$x^\mu \to \tilde{x}^\mu \simeq x^\mu + \xi^\mu_{(1)} + \frac{1}{2}\left(\xi^\nu_{(1)}\partial_\nu \xi^\mu_{(1)} + \xi^\mu_{(2)}\right)$$

- Then compute [Bruni, Matarrese, Mollerach, Sonego, Class. Quant. Grav. 14 (1997), 2585-2606]

$$\tilde{f}^{(1)}_{\mu\nu} = f^{(1)}_{\mu\nu} - \mathcal{L}_{\xi_{(1)}}\bar{f}_{\mu\nu}\,, \qquad \tilde{f}^{(2)}_{\mu\nu} = f^{(2)}_{\mu\nu} - \mathcal{L}_{\xi_{(1)}} f^{(1)}_{\mu\nu} + \frac{1}{2}\left(\mathcal{L}^2_{\xi_{(1)}}\bar{f}_{\mu\nu} - \mathcal{L}_{\xi_{(2)}}\bar{f}_{\mu\nu}\right)$$

where $\mathcal{L}_{\xi_{(n)}}$ are the Lie derivatives along the gauge modes.

Map between perturbed FLRW and LC metrics

★ The following diagram commutes:

bkg.

$$f_{\mu\nu} = \frac{\partial y^\rho}{\partial x^\mu}\frac{\partial y^\sigma}{\partial x^\nu} g_{\rho\sigma}$$

bkg.+ perturbations

$$\begin{tikzcd} \mathrm{SPT} \arrow[r, "\text{gauge}"] \arrow[d, "\text{diff.}"'] & \widetilde{\mathrm{SPT}} \arrow[d, "\text{diff.}"] \\ \mathrm{GLC} \arrow[r, "\text{gauge}"] & \widetilde{\mathrm{GLC}} \end{tikzcd}$$

$$y^\mu \to \tilde{y}^\mu = y^\mu + \epsilon^\mu_{(1)} + \frac{1}{2}(\epsilon^\nu_{(1)}\partial_\nu \epsilon^\mu_{(1)} + \epsilon^\mu_{(2)})$$

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu_{(1)} + \frac{1}{2}(\xi^\nu_{(1)}\partial_\nu \xi^\mu_{(1)} + \xi^\mu_{(2)})$$

SVT-SPS dictionary

- ⋆ Use the fully non-linear relations

$$\begin{cases} a^2 L = -2\left(\phi - \frac{1}{2}\mathcal{C}_{rr} - \mathcal{B}_r\right) \\ aM = -\mathcal{B}_{rr} - \mathcal{C}_{rr} \\ N = \mathcal{C}_{rr} \\ aV_a = -\mathcal{B}_a - \mathcal{C}_{ra} \\ U_a = \mathcal{C}_{ra} \\ \delta\gamma_{ab} = \mathcal{C}_{ab} \end{cases} \quad \Rightarrow \quad \begin{cases} \phi = -\frac{1}{2}(a^2 L + N + 2aM) \\ \mathcal{B}_r = -N - aM \\ \mathcal{C}_{rr} = N \\ \mathcal{B}_a = -U_a - aV_a \\ \mathcal{C}_{ra} = U_a \\ \mathcal{C}_{ab} = \delta\gamma_{ab} \end{cases}$$

  to connect SVT perturbations to SPS ones.

- ⋆ Example:

$$\xi^\mu_{(n)} = \frac{\partial x^\mu}{\partial y^\nu}\epsilon^\nu_{(n)}$$

$$\tilde{\phi}^{(1)} = -\frac{1}{2}(a^2\tilde{L}^{(1)} + \tilde{N}^{(1)} + 2a\tilde{M}^{(1)}) = -\frac{1}{2}(a^2 L^{(1)} + N^{(1)} + 2aM^{(1)}) - \left(\partial_\tau + \frac{\partial_w}{a}\right)\xi^0_{(1)} = \phi^{(1)} - \frac{\partial_\eta(a\epsilon^\eta_{(1)})}{a}$$

Key advantages of the GLC gauge

Non-perturbative framework

Obtain fully non-linear expressions for light-like cosmological observables

Perturbation theory

Observables are factorized as products of perturbations evaluated at the source and observer position (**no** nested integrals along the l.o.s.)

Obs. and source are connected through $\tilde{\theta}^a = \text{const.}$ geodesics on a $w = \text{const.}$ past light-cone of a free-falling obs.

The observed redshift

- In the GLC gauge we have [Gasperini, Marozzi, Nugier, Veneziano, JCAP, 1107 (2011) 008]

$$k_\mu = \partial_\mu w \,, \qquad\qquad u_\mu = -\partial_\mu \tau$$

- Then, we compute the redshift at any order in perturbation theory:

$$\boxed{1 + z = \frac{(k^\mu u_\mu)_{\rm s}}{(k^\mu u_\mu)_{\rm o}} = \frac{\Upsilon(\tau_{\rm s}, w, \tilde{\theta}^a_{\rm s})}{\Upsilon(\tau_{\rm o}, w, \tilde{\theta}^a_{\rm o})}}$$

$$w_{\rm s} = w_{\rm o} \equiv w$$

The Jacobi map (I)

- Take the geodesic deviation equation (with λ the affine parameter along the curve):

$$\nabla_\lambda^2 \xi^\mu = R_{\alpha\beta\nu}{}^\mu k^\alpha k^\nu \xi^\beta \qquad\qquad \nabla_\lambda \equiv k^\alpha \nabla_\alpha$$

- Project (for $A = 1, 2$):

$$\xi^\mu = \xi^A s^\mu_A \qquad\qquad \xi^A = \xi^\mu s^A_\mu = g_{\mu\nu} \xi^\mu s^\nu_A$$

where

$$g_{\mu\nu} s^\mu_A s^\nu_B = \delta_{AB}\,, \qquad s^\mu_A u_\mu = 0 = s^\mu_A k_\mu = \Pi^\mu_\nu \nabla_\lambda s^\nu_A$$

$$\Pi^\mu_\nu \equiv \delta^\mu_\nu - \frac{k^\mu k_\nu}{(u^\rho k_\rho)^2} - \frac{k^\mu u_\nu + u^\mu k_\nu}{u^\rho k_\rho}$$

The Jacobi map (II)

- We obtain

$$\frac{\mathrm{d}^2\xi^A}{\mathrm{d}\lambda^2} = R^A_B \xi^B \qquad\qquad R^A_B \equiv R_{\alpha\beta\nu\mu} k^\alpha k^\nu s^\beta_B s^\mu_A$$

where

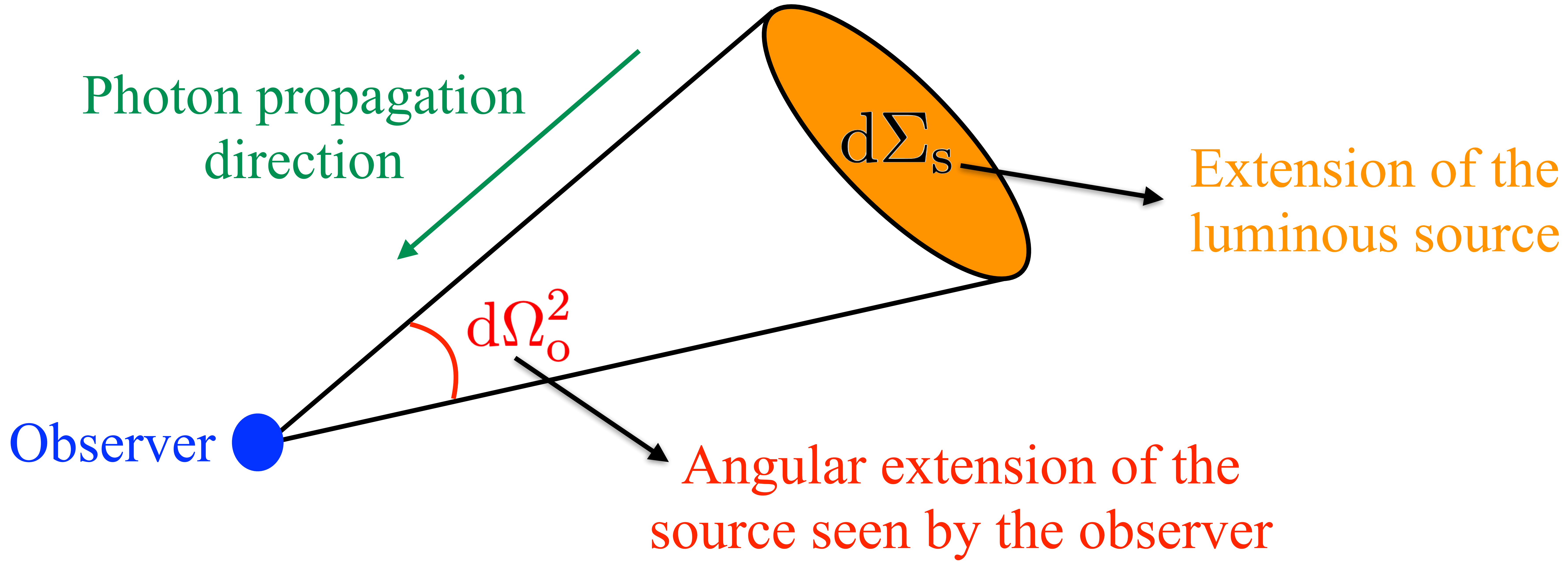
$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \equiv k^\mu \partial_\mu \boxed{= k^\tau \partial_\tau}$$

in the GLC gauge

- The Jacobi map connects an observer to a source and it is the solution to the above equation written as [Schneider, Ehlers, Falco, "Gravitational Lenses", 1992]

$$\xi^A(\lambda_\mathrm{s}) = J^A_B(\lambda_\mathrm{s}, \lambda_\mathrm{o}) \left(\frac{k^\mu \partial_\mu \xi^B}{k^\nu u_\nu}\right)_\mathrm{o}$$

The angular distance-redshift relation



In the GLC gauge
[Fanizza, Gasperini, Marozzi, Veneziano, JCAP, 1311 (2013) 019]

$$d_A^2 \equiv \frac{d\Sigma_s}{d\Omega_o^2} = \det[J_B^A] = \frac{\sqrt{\gamma}}{\left(\frac{\det[\dot{\gamma}_{ab}]}{4\sqrt{\gamma}}\right)_o}$$

Working method (I)

★ Decompose the angular gauge mode in terms of SPS gauge modes:

$$\xi^a_{(n)} = q^{ab}\left(D_b\chi_{(n)} + \tilde{D}_b\hat{\chi}_{(n)}\right), \qquad D_b\hat{\chi}_{(n)} = 0$$

★ First fix the GLC gauge on the light-cone order by order in perturbation theory:

$$\tilde{L}^{(1)} = 0 = \tilde{v}^{(1)} = \tilde{\hat{v}}^{(1)} = \tilde{N}^{(1)} + 2a\tilde{M}^{(1)}$$

$$\tilde{L}^{(2)} = 0 = \tilde{v}^{(2)} = \tilde{\hat{v}}^{(2)} = 4\tilde{N}^{(2)} - (\tilde{N}^{(1)})^2 + 8a\tilde{M}^{(2)} - 4(\tilde{U}^{(1)})^2$$

★ For example, the first order gauge modes are

$$\xi^{\tau}_{(1)} = -\frac{1}{2}\int_{\tau_{\rm in}}^{\tau} \mathrm{d}\tau' \left(a^2 L^{(1)} + N^{(1)} + 2aM^{(1)}\right)\left(\tau', w - \eta(\tau) + \eta(\tau')\right) \ ,$$

$$\xi^{w}_{(1)} = \frac{1}{2}\int_{\tau}^{\tau_o} \mathrm{d}\tau' a L^{(1)} + \boxed{w_0^{(1)}(w, \tilde{\theta}^a)} ,$$

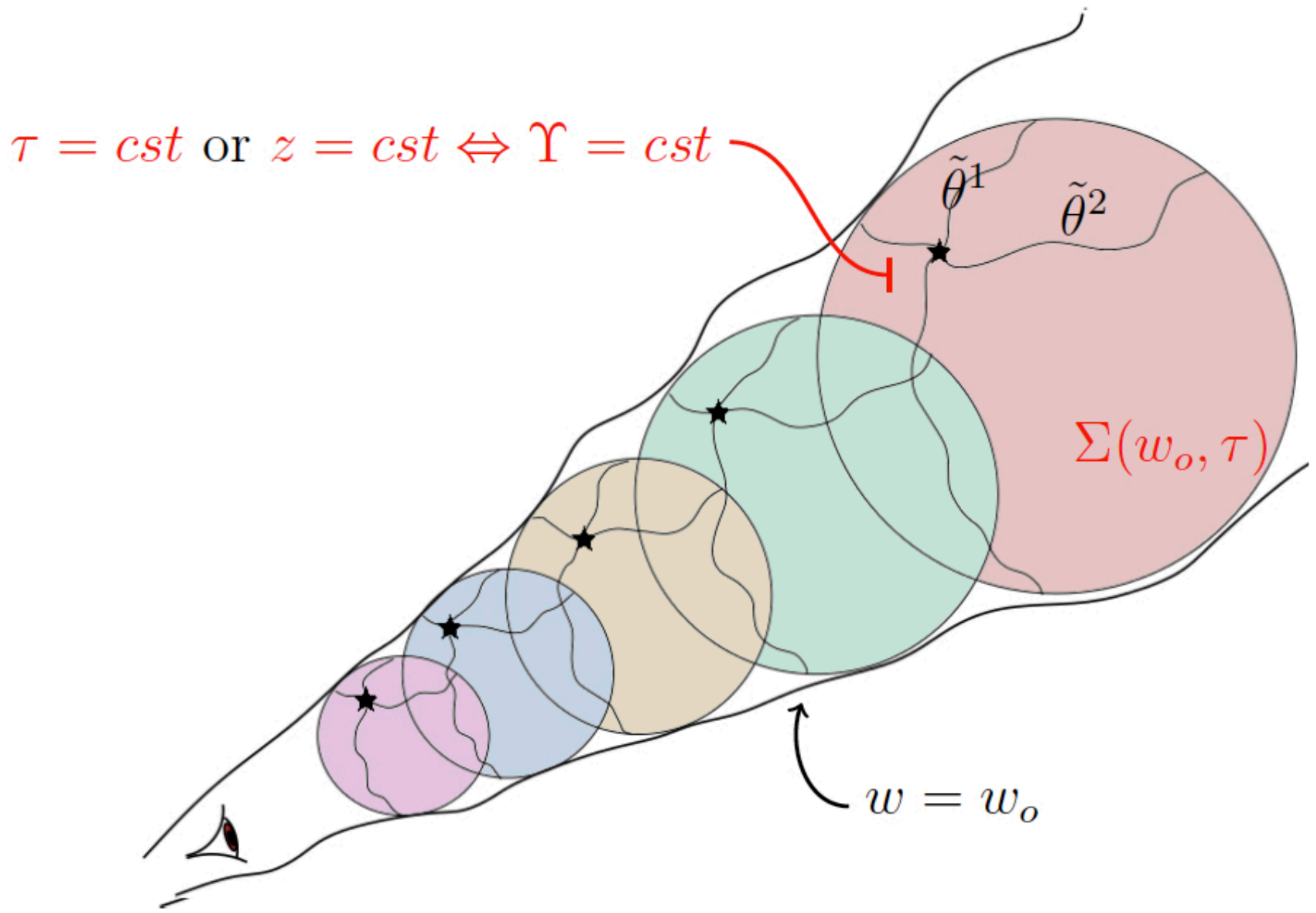
$$\chi_{(1)} = -\int_{\tau}^{\tau_o} \mathrm{d}\tau' \left(v^{(1)} + \frac{1}{2ar^2}\int_{\tau'}^{\tau_0} \mathrm{d}\tau'' a L^{(1)} + \frac{w_0^{(1)}}{ar^2}\right) + \boxed{\chi_0^{(1)}(w, \tilde{\theta}^a)} ,$$

$$\hat{\chi}_{(1)} = -\int_{\tau}^{\tau_o} \mathrm{d}\tau' \, \hat{v}^{(1)} + \boxed{\hat{\chi}_0^{(1)}(w, \tilde{\theta}^a)}$$

$$w \to w' = w'(w)$$
$$\tilde{\theta}^a \to \tilde{\theta}^{a\prime} = \tilde{\theta}^{a\prime}(w, \tilde{\theta}^a)$$

residual gauge freedom of the GLC gauge at the obs.

- “Unfix” the gauge replacing each perturbation with its gauge invariant counterpart (their value in the GLC gauge).



[Nugier, 1508.07464]

Gauge invariant expressions for light-like cosmological observables

observations by a free-falling observer

GR effects at first and second order

The Observational Synchronous Gauge

- ★ Look for the GLC gauge in terms of SPT
- ★ At first order [Fanizza, Marozzi, Medeiros, Schiaffino, JCAP, 02 (2021) 014]

$$\boxed{g_{00}^{(1)} \propto \phi^{(1)} = 0 \quad \Rightarrow \quad \text{Observational Synchronous Gauge}}$$

$\tilde{\theta}^a$ directions of observation in the sky

a geodesic obs. measures an unperturbed time

- ★ At second order

$$\boxed{\phi^{(2)} = -\frac{1}{8}\left[\left(N^{(1)}\right)^2 + 4\bar{\gamma}^{ab}U_a^{(1)}U_b^{(1)}\right] \neq 0}$$

Why?

inhomogeneities in the sky $\longleftrightarrow$ $\tilde{\theta}^a_{\text{obs.}} = \tilde{\theta}^a_{\text{source}}$

$\updownarrow$ $\qquad\qquad$ $\updownarrow$

perturbations in g_{0i} $\longleftrightarrow$ perturbations in g_{00}

Prescription at the observer position

d_{A} is a bi-scalar

Fix the gauges both at the observer and source position

see later…

$$r_{\mathrm{o}} = w_{\mathrm{o}} - \eta(\tau_{\mathrm{o}}) = 0$$

observer sit at the center of the polar frame

Preserve the "observational gauge"

$$(\xi^{w}_{(1)})_{\mathrm{o}} \equiv w_0^{(1)} = (\epsilon^{\eta}_{(1)})_{\mathrm{o}} = \frac{(\xi^{\tau}_{(1)})_{\mathrm{o}}}{a_{\mathrm{o}}}$$

No angular dependence at the observer position

$$\chi_0^{(1)} = \chi_0^{(1)}(w)\,, \qquad \hat{\chi}_0^{(1)} = \hat{\chi}_0^{(1)}(w)$$

Elimination of divergences

- With the gauge fixing of the residual GLC gauge freedom

$$w_0^{(1)} = \frac{(\xi^\tau_{(1)})_o}{a_o}\,, \qquad w_0^{(2)} = \frac{(\xi^\tau_{(2)})_o}{a_o}\,,$$

$$\chi_0^{(1)} = \chi_0^{(1)}(w)\,, \qquad \hat{\chi}_0^{(1)} = \hat{\chi}_0^{(1)}(w)$$

we are able to eliminate all the terms $\sim 1/r^n$, which would be IR divergences at the observer position.

Prescription at the source position (I)

- In Cosmology, we do not observe time but redshift.
- Expand the redshift in perturbation theory:

$$1+z=\frac{a_{\rm o}}{a_{\rm s}}\left\{1+\Upsilon^{(1)}|_{\rm s}^{\rm o}+\Upsilon^{(2)}|_{\rm s}^{\rm o}+(\Upsilon^{(1)}_{\rm s})^2-\Upsilon^{(1)}_{\rm o}\Upsilon^{(1)}_{\rm s}\right\}$$

$$=\frac{a_{\rm o}}{a_{\rm s}}\left\{1+\frac{N^{(1)}|_{\rm s}^{\rm o}}{2}+\frac{N^{(2)}|_{\rm s}^{\rm o}}{2}-\frac{(N^{(1)})^2|_{\rm s}^{\rm o}}{8}+\left(\frac{N^{(1)}_{\rm s}}{2}\right)^2-\frac{N^{(1)}_{\rm o}N^{(1)}_{\rm s}}{4}-\frac{1}{2}[U^2_{(1)}]_{\rm s}^{\rm o}\right\}$$

- Then expand

$$\tau=\tau_z+\tau_z^{(1)}+\tau_z^{(2)}$$

proper time of the source
evaluated at the observed redshift

distorsions due to inhomogeneities
between source and observer

Prescription at the source position (II)

- Each quantity at the source is expressed as

$$X(x^\mu) = \bar{X}_z + X_z^{(1)} + X_z^{(2)}$$

$$\bar{X}_z \equiv \bar{X}(\tau_z)\,,$$

$$X_z^{(1)} \equiv \dot{\bar{X}}(\tau_z)\tau_z^{(1)} + X^{(1)}(\tau_z)\,,$$

$$X_z^{(2)} \equiv \dot{\bar{X}}(\tau_z)\tau_z^{(2)} + \frac{1}{2}\ddot{\bar{X}}(\tau_z)(\tau_z^{(1)})^2 + \dot{X}^{(1)}(\tau_z)\tau_z^{(1)}$$

- Require $1 + z = a_\text{o}/a_z$ and find

$$\tau_z^{(1)} = \frac{1}{2H_z} N^{(1)}|_z^\text{o}\,,$$

$$\tau_z^{(2)} = \frac{1}{2H_z}\left\{N^{(2)}|_z^\text{o} - \frac{(N_\text{o}^{(1)})^2}{4} + \frac{3}{4}(N_z^{(1)})^2 - \frac{1}{2}N_\text{o}^{(1)}N_z^{(1)} - [U_{(1)}^2]_z^\text{o} + \frac{1}{4}\frac{q_z}{H_z^2}(N^{(1)}|_z^\text{o})^2 - \frac{N^{(1)}|_z^\text{o}\dot{N}_z^{(1)}}{4H_z}\right\}$$

Results for the angular distance

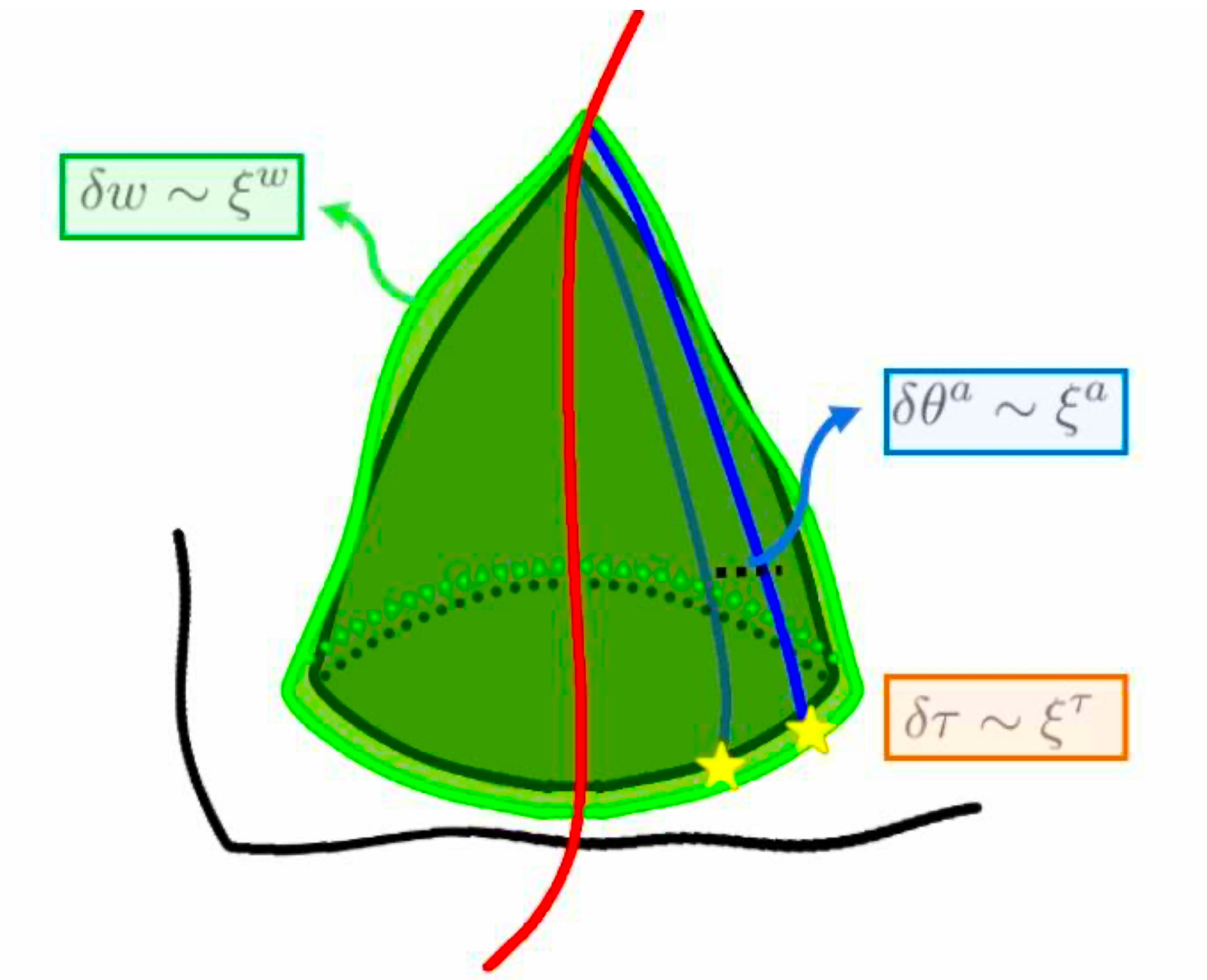
★ Fully gauge invariant formulae for the angular distance on the past light-cone:

$$d_{\rm A}^{(1)}(z) = -\mathscr{V}^{(1)}|_z^{\rm o} + \frac{1}{2}\left(1 - \frac{1}{a_z H_z r_z}\right)\mathscr{N}^{(1)}|_z^{\rm o} + (ar\dot{\mathscr{V}}^{(1)})_{\rm o}\,,$$

$$\begin{aligned} d_{\rm A}^{(2)}(z) = &-\mathscr{V}^{(2)}|_z^{\rm o} + \frac{1}{2}\left(1 - \frac{1}{a_z H_z r_z}\right)\mathscr{N}^{(2)}|_z^{\rm o} + (ar\dot{\mathscr{V}}^{(2)})_{\rm o} + \frac{1}{2}(\mathscr{V}^{(1)})^2|_z^{\rm o} + (\mathscr{V}_{\rm o}^{(1)})^2 + \frac{1}{4}\mathfrak{g}_{\rm o}^{(2)} + \\ &+ \frac{1}{4}\mathfrak{g}_z^{(2)} - \frac{1}{2}\tilde{\mathfrak{g}}_{\rm o}^{(2)} + \frac{1}{2H_z}\dot{\mathscr{V}}_z^{(1)}\mathscr{N}^{(1)}|_z^{\rm o} - \mathscr{V}_{\rm o}^{(1)}\mathscr{V}_z^{(1)} + (ar\dot{\mathscr{V}}^{(1)})_{\rm o}\left[\mathscr{V}_z^{(1)} + (ar\dot{\mathscr{V}}^{(1)})_{\rm o} - 3\mathscr{V}_{\rm o}^{(1)}\right] + \\ &+ \frac{1}{2}\left(1 - \frac{1}{a_z H_z r_z}\right)\Bigg[-\mathscr{N}^{(1)}|_z^{\rm o}\mathscr{V}^{(1)}|_z^{\rm o} - \frac{1}{4}(\mathscr{N}_{\rm o}^{(1)})^2 + \frac{3}{4}(\mathscr{N}_z^{(1)})^2 - \frac{1}{2}\mathscr{N}_{\rm o}^{(1)}\mathscr{N}_z^{(1)} + \\ &- \left[\bar{\gamma}^{ab}(D_a\mathscr{U}^{(1)} + \tilde{D}_a\hat{\mathscr{U}}^{(1)})(D_b\mathscr{U}^{(1)} + \tilde{D}_b\hat{\mathscr{U}}^{(1)})\right]_z^{\rm o} + \\ &+ \mathscr{N}^{(1)}|_z^{\rm o}(ar\dot{\mathscr{V}}^{(1)})_{\rm o}\frac{1}{4H_z}\mathscr{N}^{(1)}|_z^{\rm o}\dot{\mathscr{N}}_z^{(1)}\Bigg] + \frac{\dot{H}_z}{8a_z H_z^3 r_z}(\mathscr{N}^{(1)}|_z^{\rm o})^2 \end{aligned}$$

General relativistic effects

- ★ To compare with the literature, we evaluate the angular distance in the Newtonian gauge.
- ★ Each gauge mode needed to fix the gauge corresponds to a physical effect.
- ★ For example, at first order we have



[Courtesy of M. Medeiros]

$$\xi^\tau_{(1,\mathrm{NG})} = \overbrace{\boxed{\int_{\eta_{\mathrm{in}}}^{\eta} \mathrm{d}\eta' a \Phi^{(1)}}}^{\text{velocity potential}},$$

$$\xi^w_{(1,\mathrm{NG})} = \boxed{-\int_{\eta}^{\eta_o} \mathrm{d}\eta' (\Phi^{(1)} + \Psi^{(1)})} + w_0^{(1)}(w)\,,$$

$\partial_w \xi^w \longrightarrow$ local and integrated SW

$$D_a \xi^a_{(1,\mathrm{NG})} = \underbrace{\boxed{\int_{\eta}^{\eta_o} \mathrm{d}\eta' \int_{\eta'}^{\eta_o} \mathrm{d}\eta'' \frac{1}{r^2} D^2 (\Phi^{(1)} + \Psi^{(1)})}}_{\text{lensing}}$$

Angular distance at first order in Newtonian gauge

$$
\begin{aligned}
d_{\rm A}^{(1,\rm NG)}(z) = &-\Psi_z^{(1)} + \left(1 - \frac{1}{r_z\mathcal{H}_z}\right)\Phi^{(1)}|_z^{\rm o} + \\
&- \frac{1}{2}\int_{\eta_z}^{\eta_{\rm o}} \frac{{\rm d}\eta}{r^2}\int_{\eta}^{\eta_{\rm o}} {\rm d}\eta' D^2(\Phi^{(1)} + \Psi^{(1)}) + \frac{1}{r_z}\int_{\eta_z}^{\eta_{\rm o}} {\rm d}\eta\,(\Phi^{(1)} + \Psi^{(1)}) + \\
&- \left(1 - \frac{1}{r_z\mathcal{H}_z}\right)\left(\int_{\eta_z}^{\eta_{\rm o}} {\rm d}\eta\,\partial_\eta(\Phi^{(1)} + \Psi^{(1)}) + \frac{1}{a_z}\int_{\eta_{\rm in}}^{\eta_z} {\rm d}\eta\, a\partial_r\Phi^{(1)}\right) + \\
&- \frac{1}{a_{\rm o} r_z\mathcal{H}_z}\int_{\eta_{\rm in}}^{\eta_{\rm o}} {\rm d}\eta\, a\partial_r\Phi^{(1)} \boxed{- \left(\mathcal{H}_{\rm o} - \frac{\mathcal{H}_{\rm o}}{r_z\mathcal{H}_z} + \frac{1}{r_z}\right)\frac{1}{a_{\rm o}}\int_{\eta_{\rm in}}^{\eta_{\rm o}} {\rm d}\eta\, a\Phi^{(1)}}
\end{aligned}
$$

extra term at the obs. such that $d_{\rm A}(z)$ is the one **measured by a free-falling obs**.

Summary and Outlook

* **Summary:**
  * We developed the formalism for LC gauge invariant variables and observables beyond linear order.
  * At second order, we get a very long formula for $d_{\mathrm{A}}(z)$, with all the various GR effects and the second order correction at the observer (new terms not present in the literature).
  * The full control of observer terms at second order provides the gauge invariant formula for $d_{\mathrm{A}}(z)$ beyond linear order and as seen by a free-falling observer.

* **Outlook:**
  * These new tools can be conveniently applied to compute other cosmological observables on the light-cone up to second order (*e.g.* the redshift drift).

Thanks for your attention!

Back-up slides

SVT-SPS explicit dictionary

$$L = -\frac{2}{a^2}\left[\phi + \psi - \left(\nabla_r\nabla_r - \frac{1}{3}\Delta_3\right)E - \partial_r B - B_r - \nabla_r F_r - h_{rr}\right],$$

$$M = -\frac{1}{a}\left[-2\psi + 2\left(\nabla_r\nabla_r - \frac{1}{3}\Delta_3\right)E + \partial_r B + B_r + 2\nabla_r F_r + 2h_{rr}\right],$$

$$N = -2\psi + 2\left(\nabla_r\nabla_r - \frac{1}{3}\Delta_3\right)E + 2\nabla_r F_r + 2h_{rr}\,,$$

$$v = -\frac{1}{a}\frac{1}{D^2}\left[\bar{\gamma}^{ab}D_{(a}\left(2\nabla_{(r}\nabla_{b))}E + \partial_{b)}B + B_{b)} + 2\nabla_{(r}F_{b))} + 2h_{rb)}\right)\right],$$

$$\hat{v} = -\frac{1}{a}\frac{1}{D^2}\left[\bar{\gamma}^{ab}\tilde{D}_{(a}\left(2\nabla_{(r}\nabla_{b))}E + \partial_{b)}B + B_{b)} + 2\nabla_{(r}F_{b))} + 2h_{rb)}\right)\right],$$

$$u = \frac{2}{D^2}\left[\bar{\gamma}^{ab}D_{(a}\left(\nabla_{(r}\nabla_{b))}E + \nabla_{(r}F_{b))} + h_{rb)}\right)\right],$$

$$\hat{u} = \frac{2}{D^2}\left[\bar{\gamma}^{ab}\tilde{D}_{(a}\left(\nabla_{(r}\nabla_{b))}E + \nabla_{(r}F_{b))} + h_{rb)}\right)\right],$$

$$\nu = -\left(\psi + \frac{1}{3}\Delta_3 E\right) + \frac{\bar{\gamma}^{ab}}{2}\left[\nabla_{(a}\left(\nabla_{b)}E + F_{b)}\right) + h_{ab}\right],$$

$$\mu = \frac{2}{r^2}\frac{1}{D^2(D^2+2)}\left[D^{ab}\left(\nabla_{(a}(\nabla_{b)}E + F_{b)}) + h_{ab}\right)\right],$$

$$\hat{\mu} = \frac{2}{r^2}\frac{1}{D^2(D^2+2)}\left[\tilde{D}^{ab}\left(\nabla_{(a}(\nabla_{b)}E + F_{b)}) + h_{ab}\right)\right].$$

SVT-SPS explicit dictionary

[Fanizza, Marozzi, Medeiros, JCAP, 02 (2023) 015]

$$L=-\frac{2}{a^2}\left[\phi+\psi-\left(\nabla_r\nabla_r-\frac{1}{3}\Delta_3\right)E-\partial_r B-B_r-\nabla_r F_r-h_{rr}\right],$$

$$M=-\frac{1}{a}\left[-2\psi+2\left(\nabla_r\nabla_r-\frac{1}{3}\Delta_3\right)E+\partial_r B+B_r+2\nabla_r F_r+2h_{rr}\right],$$

$$N=-2\psi+2\left(\nabla_r\nabla_r-\frac{1}{3}\Delta_3\right)E+2\nabla_r F_r+2h_{rr}\,,$$

Spin-0 perturbations

$$v=-\frac{1}{a}\frac{1}{D^2}\left[\bar\gamma^{ab}D_{(a}\left(2\nabla_{(r}\nabla_{b))}E+\partial_{b)}B+B_{b)}+2\nabla_{(r}F_{b))}+2h_{rb)}\right)\right],$$

$$\hat v=-\frac{1}{a}\frac{1}{D^2}\left[\bar\gamma^{ab}\tilde D_{(a}\left(2\nabla_{(r}\nabla_{b))}E+\partial_{b)}B+B_{b)}+2\nabla_{(r}F_{b))}+2h_{rb)}\right)\right],$$

$$u=\frac{2}{D^2}\left[\bar\gamma^{ab}D_{(a}\left(\nabla_{(r}\nabla_{b))}E+\nabla_{(r}F_{b))}+h_{rb)}\right)\right],$$

$$\hat u=\frac{2}{D^2}\left[\bar\gamma^{ab}\tilde D_{(a}\left(\nabla_{(r}\nabla_{b))}E+\nabla_{(r}F_{b))}+h_{rb)}\right)\right],$$

Spin-1 perturbations

$$\nu=-\left(\psi+\frac{1}{3}\Delta_3 E\right)+\frac{\bar\gamma^{ab}}{2}\left[\nabla_{(a}\left(\nabla_{b)}E+F_{b)}\right)+h_{ab}\right],$$

$$\mu=\frac{2}{r^2}\frac{1}{D^2(D^2+2)}\left[D^{ab}\left(\nabla_{(a}(\nabla_{b)}E+F_{b)})+h_{ab}\right)\right],$$

$$\hat\mu=\frac{2}{r^2}\frac{1}{D^2(D^2+2)}\left[\tilde D^{ab}\left(\nabla_{(a}(\nabla_{b)}E+F_{b)})+h_{ab}\right)\right].$$

Spin-2 perturbations

SVT-SPS explicit dictionary

[Fanizza, Marozzi, Medeiros, JCAP, 02 (2023) 015]

$$L = -\frac{2}{a^2}\left[\phi + \psi - \left(\nabla_r\nabla_r - \frac{1}{3}\Delta_3\right)E - \partial_r B - B_r - \nabla_r F_r - h_{rr}\right],$$

$$M = -\frac{1}{a}\left[-2\psi + 2\left(\nabla_r\nabla_r - \frac{1}{3}\Delta_3\right)E + \partial_r B + B_r + 2\nabla_r F_r + 2h_{rr}\right],$$

$$N = -2\psi + 2\left(\nabla_r\nabla_r - \frac{1}{3}\Delta_3\right)E + 2\nabla_r F_r + 2h_{rr}\,,$$

$$v = -\frac{1}{a}\frac{1}{D^2}\left[\bar{\gamma}^{ab}D_{(a}\left(2\nabla_{(r}\nabla_{b))}E + \partial_{b)}B + B_{b)} + 2\nabla_{(r}F_{b))} + 2h_{rb)}\right)\right],$$

$$\hat{v} = -\frac{1}{a}\frac{1}{D^2}\left[\bar{\gamma}^{ab}\tilde{D}_{(a}\left(2\nabla_{(r}\nabla_{b))}E + \partial_{b)}B + B_{b)} + 2\nabla_{(r}F_{b))} + 2h_{rb)}\right)\right],$$

$$u = \frac{2}{D^2}\left[\bar{\gamma}^{ab}D_{(a}\left(\nabla_{(r}\nabla_{b))}E + \nabla_{(r}F_{b))} + h_{rb)}\right)\right],$$

$$\hat{u} = \frac{2}{D^2}\left[\bar{\gamma}^{ab}\tilde{D}_{(a}\left(\nabla_{(r}\nabla_{b))}E + \nabla_{(r}F_{b))} + h_{rb)}\right)\right],$$

$$\nu = -\left(\psi + \frac{1}{3}\Delta_3 E\right) + \frac{\bar{\gamma}^{ab}}{2}\left[\nabla_{(a}\left(\nabla_{b)}E + F_{b)}\right) + h_{ab}\right],$$

$$\mu = \frac{2}{r^2}\frac{1}{D^2(D^2+2)}\left[D^{ab}\left(\nabla_{(a}(\nabla_{b)}E + F_{b)}) + h_{ab}\right)\right],$$

$$\hat{\mu} = \frac{2}{r^2}\frac{1}{D^2(D^2+2)}\left[\tilde{D}^{ab}\left(\nabla_{(a}(\nabla_{b)}E + F_{b)}) + h_{ab}\right)\right].$$

E-modes

B-modes

Free-falling observer of the OSG (I)

The condition $g^{00} = -1$ defines a free-falling obs. $\longrightarrow$ $\boxed{\partial^\nu \tau \nabla_\nu (\partial_\mu \tau) = 0}$

$$u_\mu = -\partial_\mu \tau = \delta^\tau_\mu$$

$$\partial^\nu \tau \nabla_\nu (\partial_\mu \tau) = -g^{\tau\nu}\Gamma^\tau_{\mu\nu} = -\frac{1}{2} g^{\tau\nu} g^{\tau\rho} \partial_\mu g_{\nu\rho}$$

$$\partial_\mu g^{\tau\tau} = \partial_\mu (g^{\tau\rho} g^{\tau\nu} g_{\nu\rho}) = 0 \longrightarrow g^{\tau\nu} g^{\tau\rho} \partial_\mu g_{\rho\nu} = -g^{\tau\nu} g_{\rho\nu} \partial_\mu g^{\tau\rho} - g^{\tau\rho} g_{\nu\rho} \partial_\mu g^{\tau\nu} = -2\partial_\mu g^{\tau\tau} = 0$$

Free-falling observer of the OSG (II)

- Take the inverse (0,0)-components at first and second order:

$$g^{\tau\tau}_{(1)} = -a^2 L^{(1)} - 2aM^{(1)} - N^{(1)}$$

$$g^{\tau\tau}_{(2)} = -a^2 L^{(2)} - 2aM^{(2)} - N^{(2)} - (a^2 L^{(1)})^2 - 4a^3 L^{(1)} M^{(1)} - 3a^2 (M^{(1)})^2 + \\ - 2a^2 L^{(1)} N^{(1)} - 2aM^{(1)} N^{(1)} + \bar{\gamma}^{ab} U_a^{(1)} U_b^{(1)} + a^2 \bar{\gamma}^{ab} V_a^{(1)} V_b^{(1)} + 2a\bar{\gamma}^{ab} V_a^{(1)} U_b^{(1)}$$

- Inserting the GLC gauge fixing conditions, we get:

$$g^{\tau\tau}_{(1,\mathrm{GLC})} = -2aM^{(1)} - N^{(1)} = 0$$

$$g^{\tau\tau}_{(2,\mathrm{GLC})} = -2aM^{(2)} - \frac{1}{4}(N^{(1)})^2 + 2aM^{(2)} - U^2_{(1)} - a\left[3a(M^{(1)})^2 + 2M^{(1)}N^{(1)}\right] + U^2_{(1)}$$
$$= -\frac{1}{4}(2aM^{(1)})^2 - a\left[3a(M^{(1)})^2 - 4a(M^{(1)})^2\right] = 0$$

GLC gauge invariant variables at first order

$$\mathscr{V}^{(1)} \equiv \nu^{(1)} - \frac{1}{2}D^2\chi_{(1)} - \xi^0_{(1)}\left(H - \frac{1}{ar}\right) - \frac{\xi^w_{(1)}}{r}\,,$$

$$\mathscr{N}^{(1)} \equiv N^{(1)} - 2H\xi^0_{(1)} + \frac{2}{a}\partial_w\xi^0_{(1)} - 2\partial_w\xi^w_{(1)}\,,$$

$$\mathscr{M}^{(1)} \equiv \mu^{(1)} - \chi_{(1)}\,,$$

$$\hat{\mathscr{M}}^{(1)} \equiv \hat{\mu}^{(1)} - \hat{\chi}_{(1)}\,,$$

$$\mathscr{U}^{(1)} \equiv u^{(1)} + \frac{\xi^0_{(1)}}{ar^2} - \frac{\xi^w_{(1)}}{r^2} - \partial_w\chi_{(1)}\,,$$

$$\hat{\mathscr{U}}^{(1)} \equiv \hat{u}^{(1)} - \partial_w\hat{\chi}_{(1)}\,.$$