Analytical routes to massive inflationary correlators

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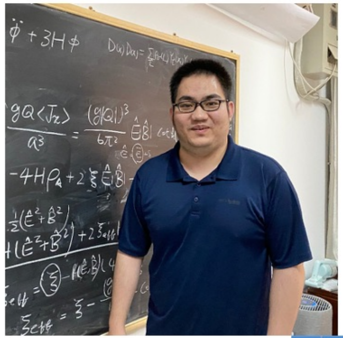


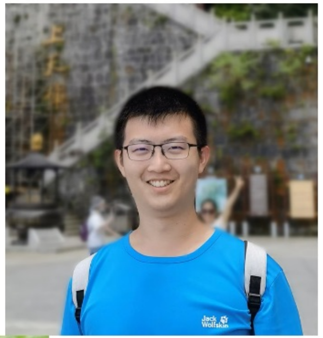


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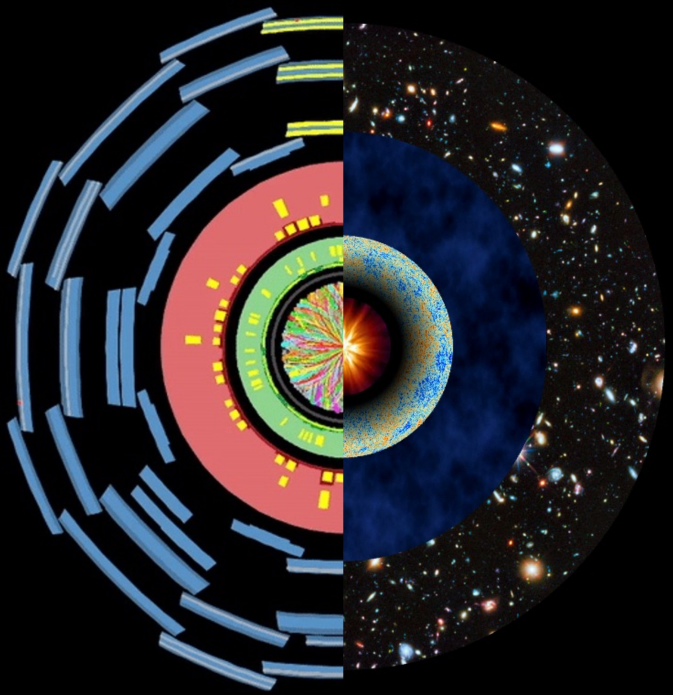
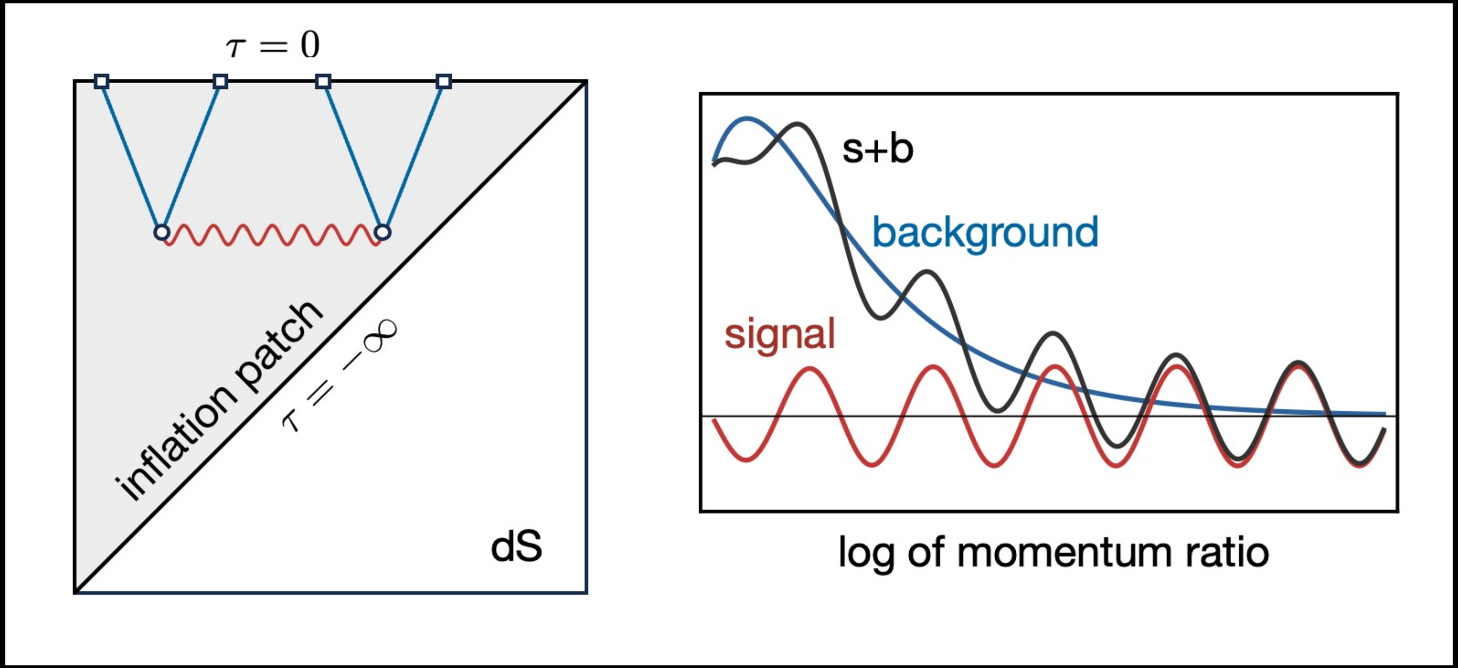
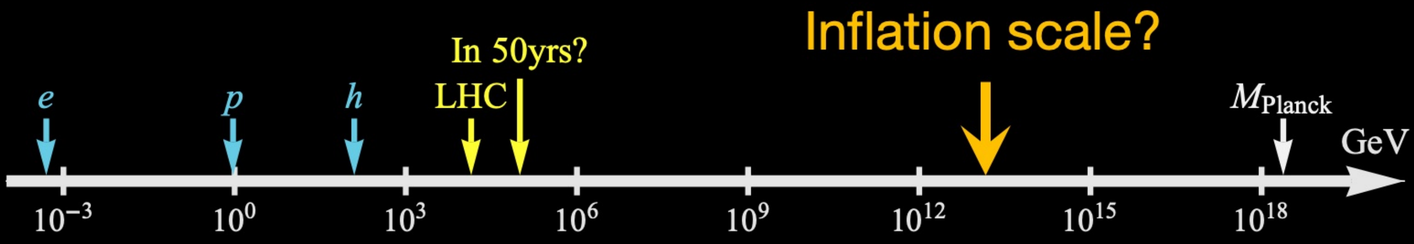
A cosmological collider program

[Chen, Wang, 0911.3380; Arkani-Hamed, Maldacena, 1503.08043 and many more]



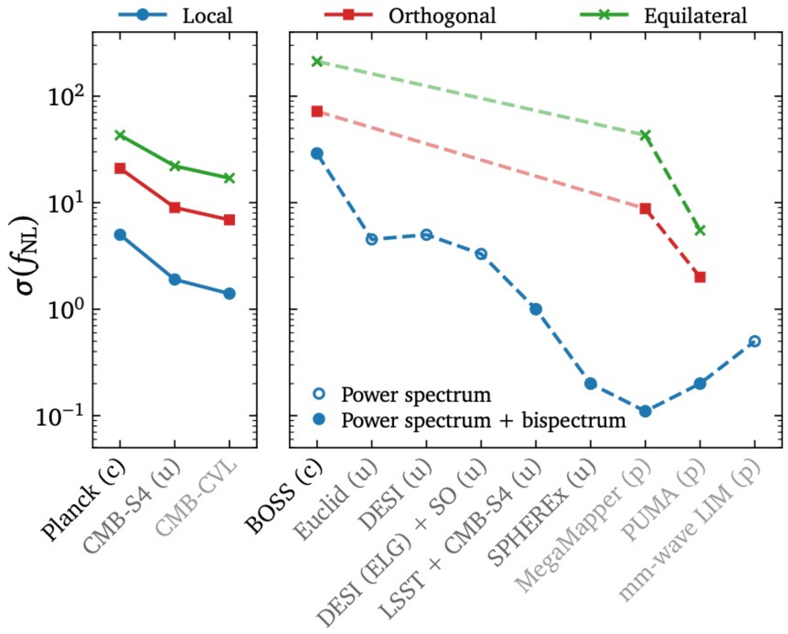




Data are coming in!

- ~ 2 orders in near future; ~ 4 ultimately with 21cm

[Snowmass 2021: 2203.08128]

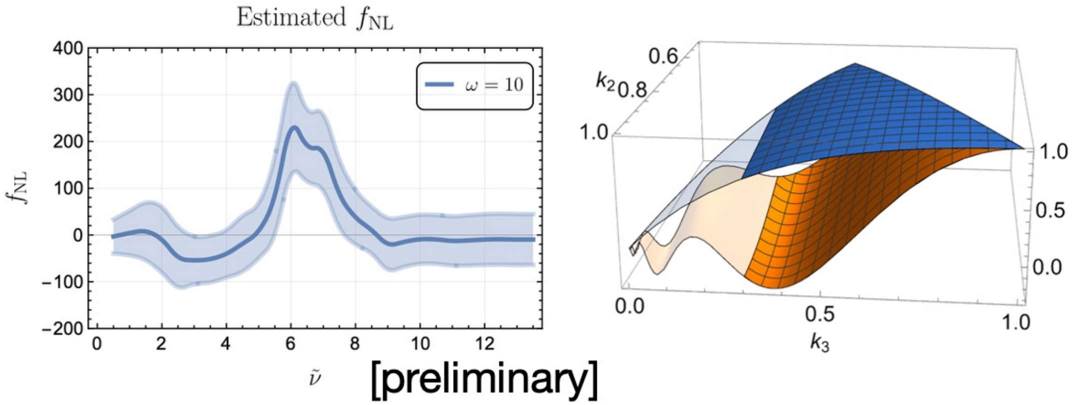


- Challenges for theorists:
  Efficient and precise computation of massive cosmological correlators!

- Searches of single-change processes from CMB/LSS data [Cabass et al. 2404.01894; Sohn et al. 2404.07203; Suman et al. 2511.17500; Philcox et al. 2511.19179 etc]

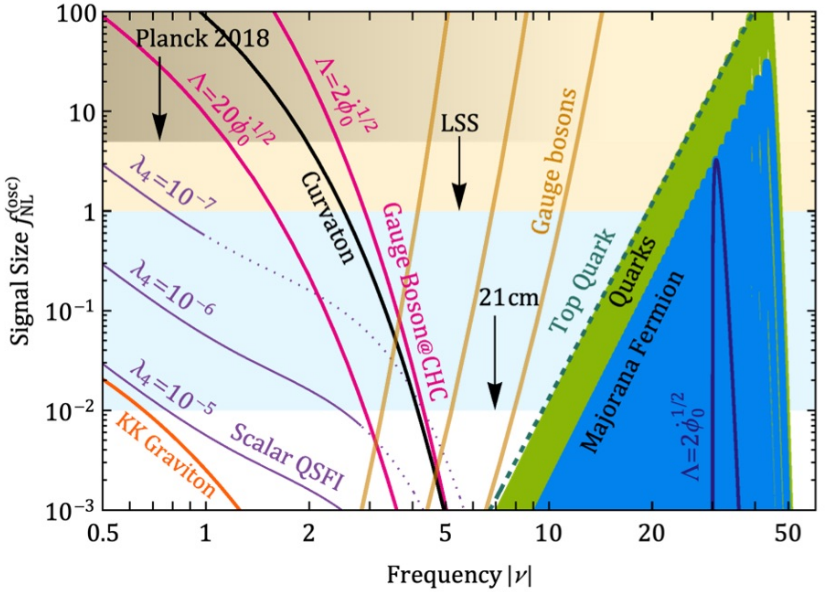
- Parity-violating particle models (chem potential) from LSS data
  [Bao, Wang, ZX, Zhong, 2504.02931]

- Many types of scalar-exchange models from Planck data [Kumar, Lu, ZX, Zhang, to appear]



[preliminary]

Particle Phenomenology



[Lian-Tao Wang, ZX, 1910.12876]

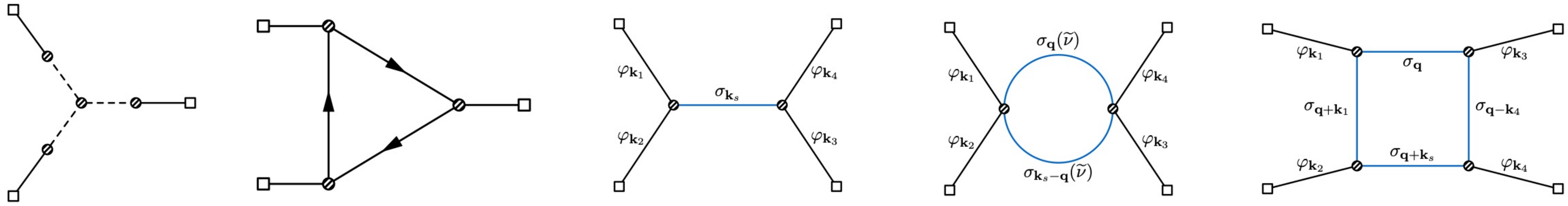
Over the years, many particle models identified in SM/BSM, with naturally large signals

Many fascinating stories which are still ongoing

The CC signals can be there, and deserve to be treated seriously

Model templates

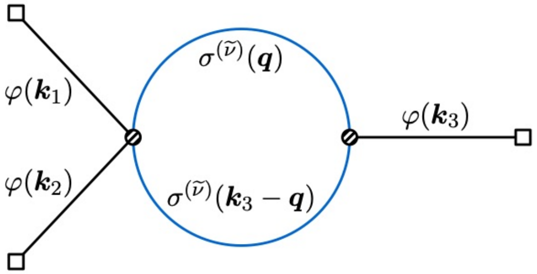
- Behind the CC signals are “simple” Feynman graphs in the inflationary background:



- To look for CC signals in real data, we may need a template bank
- Not a kinematic point, but the full shape; not for a parameter; but a multi-dim parameter grid
- We’d better compute them with precision and efficiency --- Analytic approach
- Developing fast! Many computations considered impossible a few years ago are now done
- Meanwhile, active studies on numerical frontier as well
  [Werth, Pinol, Renaux-Petel, 2302.00655, 2402.03693, 2312.06559]

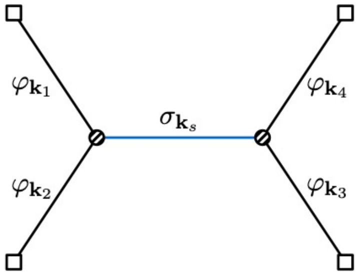
Why analytic?

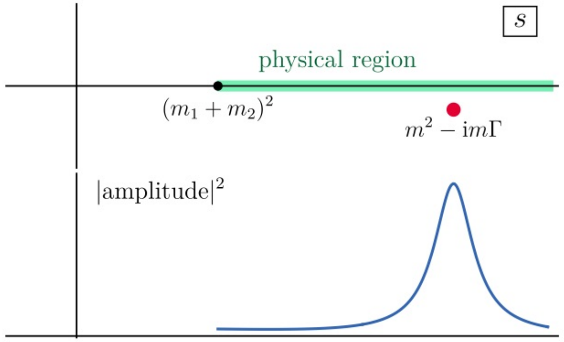
- Data-wise: good analytical strategy speeds up numerical computation
  Example: 3pt massive bubble: numerical [O(10^5) CPU hrs] vs. analytical [O(10s) @ laptop]
  [Wang, ZX, Zhong, 2109.14635] [Liu, Qin, ZX, 2407.12299]

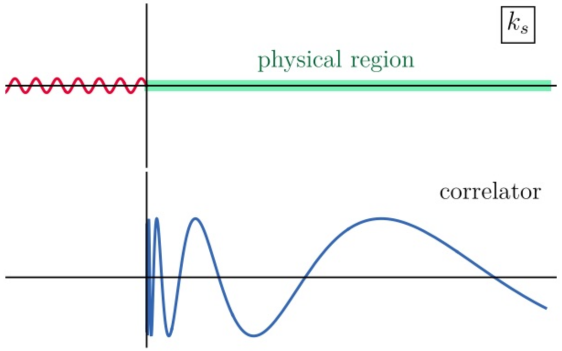


$$\mathcal{J}^{0,-2}(u) = Cu^3 - \frac{u^4}{128\pi\sin(2\pi i\tilde\nu)}\sum_{n=0}^{\infty} \frac{(3+4i\tilde\nu+4n)(1+n)_{\frac12}(1+2i\tilde\nu+n)_{\frac12}}{(\frac12+i\tilde\nu+n)_{\frac12}(\frac32+i\tilde\nu+n)_{\frac12}}$$
$$\times \left\{ {}_2\mathcal{F}_1\left[\begin{matrix}2+2i\tilde\nu+2n, 4+2i\tilde\nu+2n\\ 4+4i\tilde\nu+4n\end{matrix}\middle| u\right] u^{2n+2i\tilde\nu} - {}_3\mathcal{F}_2\left[\begin{matrix}1,2,4\\ 1-2n-2i\tilde\nu, 4+2n+2i\tilde\nu\end{matrix}\middle| u\right]\right\}$$
$$+ (\tilde\nu\to-\tilde\nu)$$

- Theory-wise: good lessons about QFT in dS from analytical structures of correlators
  Whenever a correlator becomes singular, there is a physical reason







[Qin, ZX, 2308.14802]

The nature of the problem

- Weakly coupled QFTs in the bulk (loop expansion works)
- dS isometries often broken (especially dS boosts, sometimes dilitation as well)
- They suggest us to develop analytical methods that exploit bulk perturbativity in a model independent way but do not heavily rely on full dS isometries
- The goals:
  1. Explicit results of correlators in terms of well-defined functions
  2. Understanding the analytic structures: complete characterization of their singularities
  3. Analytical continuation and efficient numerical strategies
- A comment on tensor structure:

$$\langle\varphi_{\boldsymbol{k}_1}\cdots\varphi_{\boldsymbol{k}_n}\rangle' \sim \sum_{\text{tensor structure}} \text{tensorial factor} \times \text{scalar integral}$$

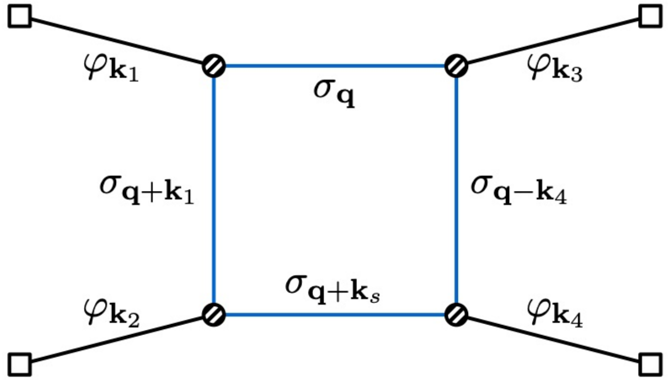
Massive inflation correlators

[See Chen, Wang, ZX, 1703.10166 for a review]

$$\mathcal{T}(\{\boldsymbol{k}\}) \sim \int \mathrm{d}\tau \int \mathrm{d}^d\boldsymbol{q} \times (-\tau)^p \times e^{\mathrm{i}E\tau} \times \mathrm{H}_{\mathrm{i}\widetilde{\nu}}\Big[-K(\boldsymbol{q},\boldsymbol{k})\tau\Big] \times \theta(\tau_i - \tau_j)$$

vertex int   loop int   ext line   bulk line

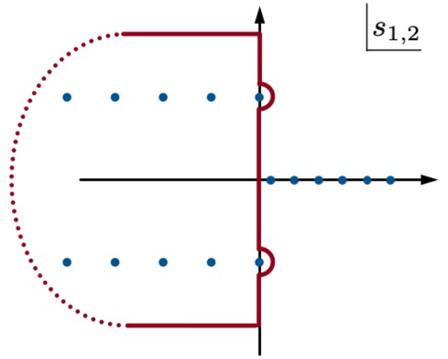
- Massless / conformal external lines + (principal) massive internal lines
- Challenges:
  - Mode functions (Hankel, Whittacker, …)
  - Nested time integrals
  - Loop momentum integrals
- Complexity increases with # of loops and # of vertices
  [Tree graphs are nontrivial! Quantifiable; more later]



Analytical methods

Expression => Structure

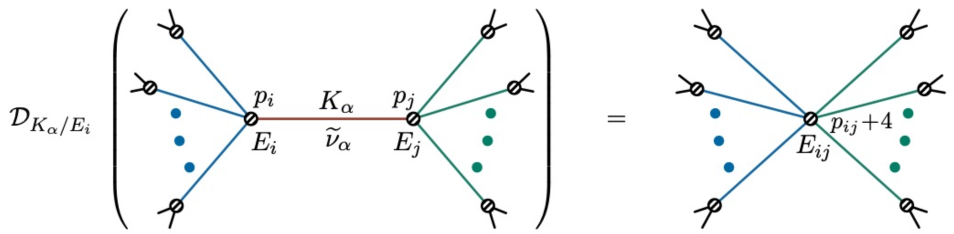
Partial Mellin-Barnes representation [Resolve!]



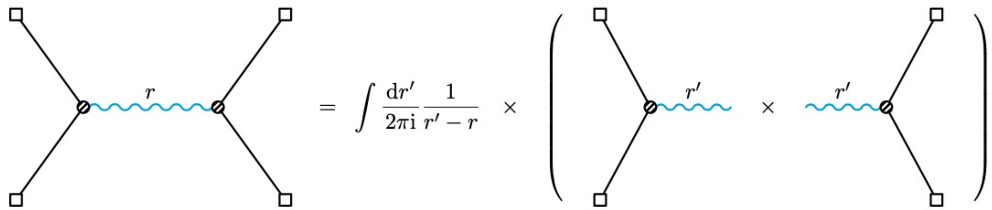
Family tree decomposition [Flip!]

Structure => Expression

Differential equations [Pinch!]



Dispersion relations [Glue!]



Partial Mellin-Barnes representation

[Qin, ZX, 2205.01692, 2208.13790]

MB rep for all bulk lines; Resolving special functions into power functions

For example: Massive scalar propagator [Hankel function]

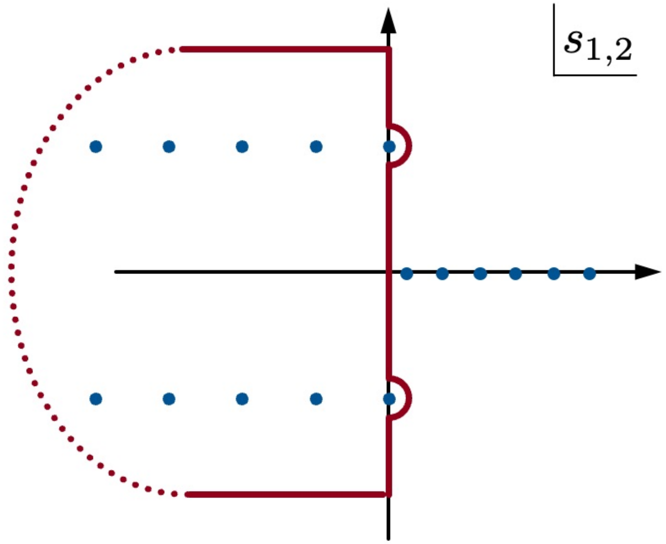
$$\mathrm{H}_\nu^{(1)}(-k\tau) = \frac{1}{\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\left(\frac{k}{2}\right)^{-2s}(-\tau)^{-2s}e^{(2s-\nu-1)\pi\mathrm{i}/2}\Gamma\left[s-\frac{\nu}{2}, s+\frac{\nu}{2}\right]$$

Expanding in dilatation eigenmode, but no dilatation or boost symmetry required

Time and loop momentum integrals factorized, enabling separate treatments

$$\mathcal{T}(\{\boldsymbol{k}\}) \sim \int \mathrm{d}s \times \mathcal{G}(s) \times \left[\int \mathrm{d}^d\boldsymbol{q} K(\boldsymbol{q},\boldsymbol{k})^\alpha\right] \quad \text{Loop int}$$

$$\times \left[\int \mathrm{d}\tau e^{\mathrm{i}E\tau} \times (-\tau)^\beta \times \theta(\tau_i-\tau_j)\right] \quad \text{Time int}$$



[See also Sleight 1907.01143 etc.]

Family tree decomposition

[ZX, Zang, 2309.10849]

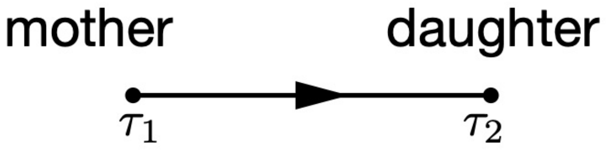
Family tree decomposition: flip the directions such that all graphs are partially ordered

$$\theta(\tau_1-\tau_2)+\theta(\tau_2-\tau_1)=1$$

Partial order:

A mother can have any number of daughters

but a daughter must have only one mother



Every resulting nested graph can be interpreted as a maternal family tree

sisters

A notation for FTs: $$\big[12(34\cdots)(5\cdots)\big]=\int_{-\infty}^{0}\prod_{i=1}^{N}\Big[\mathrm{d}\tau_i(-\tau_i)^{q_i-1}e^{\mathrm{i}\omega_i\tau_i}\Big]\theta_{21}\theta_{32}\theta_{52}\theta_{43}\cdots$$

mother-daughter

$$\big[\mathscr{P}(\widehat{1}2\cdots N)\big]=\frac{(-\mathrm{i})^N}{(\mathrm{i}\omega_1)^{q_{1\cdots N}}}\sum_{n_2,\cdots,n_N=0}^{\infty}\Gamma(q_{1\cdots N}+n_{2\cdots N})\prod_{j=2}^{N}\frac{(-\omega_j/\omega_1)^{n_j}}{(\widetilde{q}_j+\widetilde{n}_j)n_j!}$$

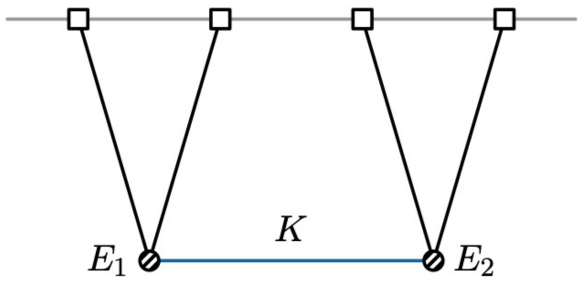
What can PMB + FTD do?

- Tree level: A trivial procedure to get analytical results for all trees [solved]
  PMB => FTD => Collecting MB poles => Solutions in hypergeo series [ZX, Zang, 2309.10849]

- A byproduct: complete analytical answer for all conformal scalar tree amplitudes in power-law FRW space [automatically solve the kinematic flow diff eqs] [Fan, ZX, 2403.07050]

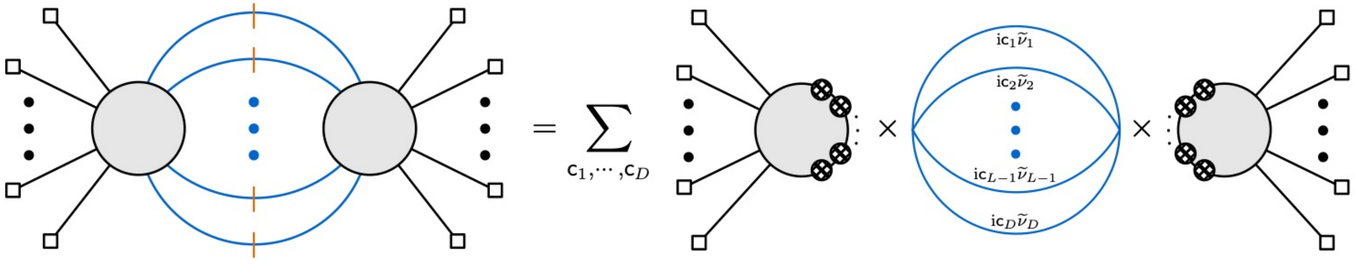
- Beyond tree level: Full computation remains challenging
  However, very useful for studying analytical structure of arbitrary loop graphs

- Generally, a (tree or loop) correlator can exhibit singular behavior (branch point) at:
  - ☐ Nonlocal signal branch points (soft momentum limit)
    $K \to 0$
  - ☐ Local signal branch points (hard energy limit)
    $E_1 \to \infty$ or $E_2 \to \infty$
  - ☐ Partial energy branch points (zero energy sum limit)
    $E_1 + K \to 0$ or $E_2 + K \to 0$ or $E_1 + E_2 \to 0$



Singularity structure / factorization theorems / cutting rules

- Take the nonlocal signal as an example (very relevant to CC pheno)
- The nonlocal signal is factorized (and thus cut) and computable to the leading order in the soft momentum but to all loop orders [Qin, ZX, 2304.13295; 2308.14802]



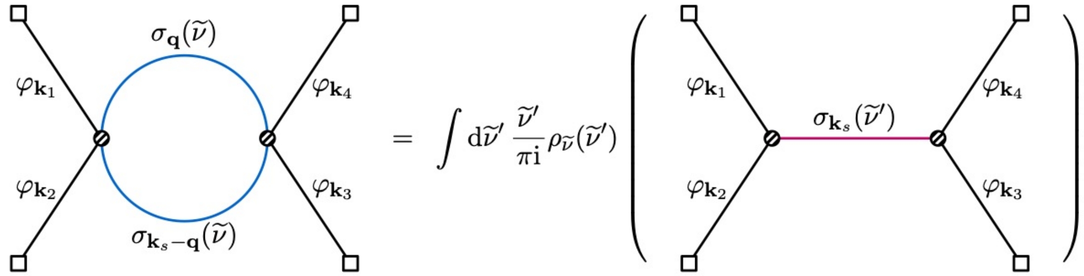
$$\mathfrak{M}_{c_1\cdots c_D}(P) \equiv \frac{P^{3(D-1)}}{(4\pi)^{(5D-3)/2}}\Gamma\begin{bmatrix} -\sum_{i=1}^{D} c_i i\widetilde{\nu}_i - \frac{3}{2}(D-1) \\ \frac{3}{2}D + \sum_{i=1}^{D} c_i i\widetilde{\nu}_i \end{bmatrix}\prod_{\ell=1}^{D}\left\{\Gamma\left[\frac{3}{2}+c_\ell i\widetilde{\nu}_\ell, -c_\ell i\widetilde{\nu}_\ell\right]\left(\frac{P}{2}\right)^{2ic_\ell\widetilde{\nu}_\ell}\right\}$$

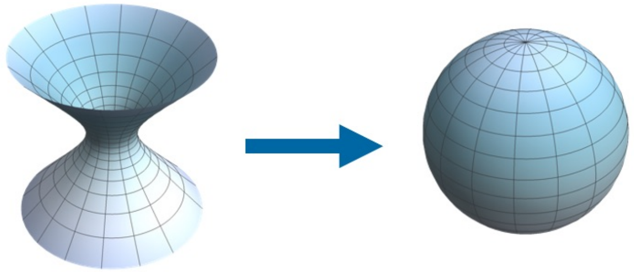
- Similar factorization and cutting rules hold for the local signal [Qin, ZX, to appear] and partial-energy limit [Wu, ZX, Zhang, to appear]
- In a sense, the nonanalytic part is always "simpler" than the analytic part

Spectral decomposition of loops

[ZX, Zhang, 2211.03810; Zhang, 2507.19318]

Loops greatly simplified with new strategies in certain cases: spectral decomposition

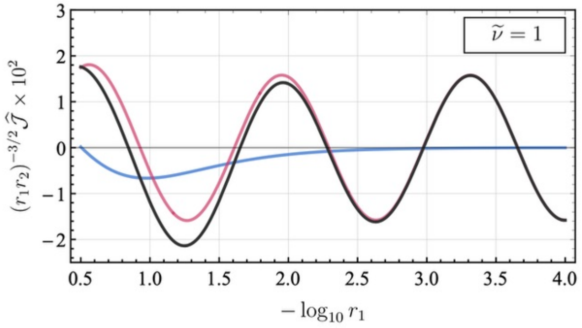


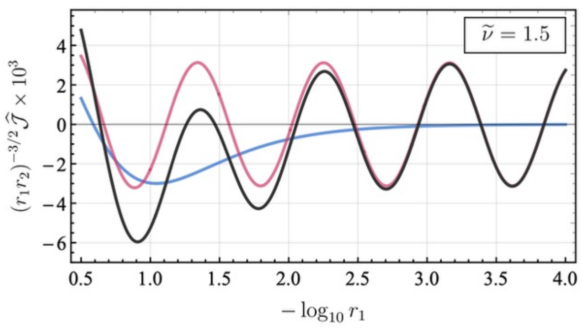


Rewrite bubble 1-loop as linear superposition of tree graphs with all possible masses.

The spectral density obtainable by Wick-rotating dS to sphere or AdS

Complete analytical expressions obtained for bubble graphs of massive scalar and vector boson exchange. Directly generalizable to other spins and banana graphs



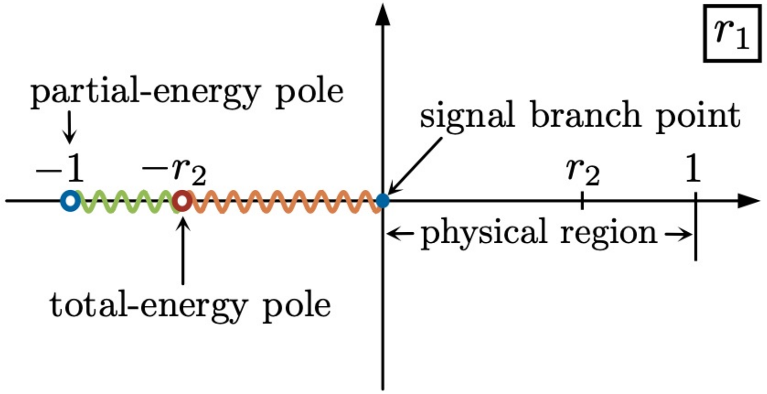


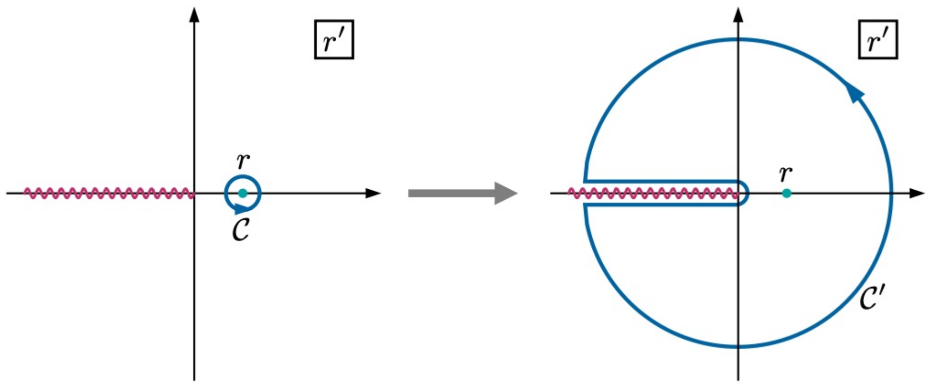
A dispersive boostrap

[Liu, Qin, ZX, 2407.12299; Liu, Qin, Wu, ZX, Zhang, to appear]

The study of analyticity allows us to locate all singularities on the complex plane

=> Bootstraping complex graphs by gluing simpler ones. The glue: dispersion integral





Dispersion integrals are insensitive to UV (local) physics

New and much simplified analytical expression for loops; UV and IR neatly separated

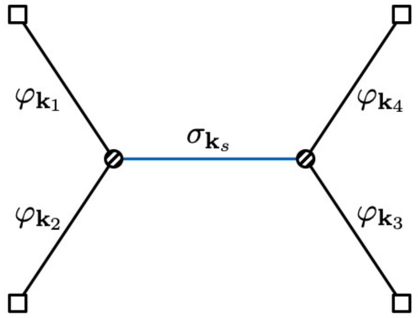
In particular: we identify an "irreducible background" demanded by analyticity

Lesson: UV div/regularization artificial and avoidable; but renormalization physical

[See also: Werth, 2409.02072]

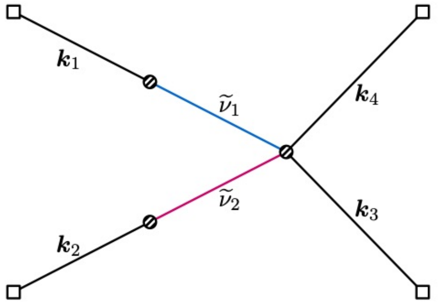
Differential equations

- “Old” technique, first used in cosmo correlators as a “bootstrap” equation
- However, much easier to derive and to generalize in the bulk



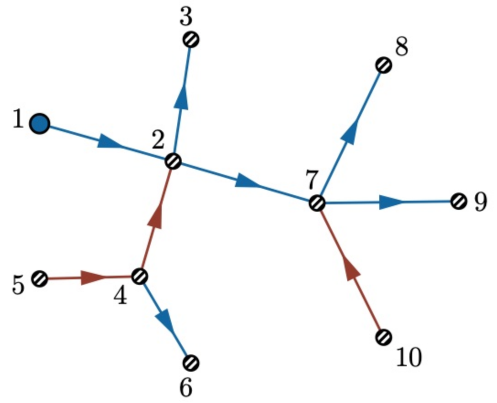
1 exchange (2018)

“Cosmological bootstrap”
[Arkani-Hamed, Baumann, Lee, Pimentel, 1811.00024]



2 exchanges (2024)

[ZX, Zang, 2309.10849]
[Aoki, Pinol, Sano, Yamaguchi, Zhu, 2404.09547]



Arbitrary exchanges (2023)

Partial Mellin-Barnes
[Qin, ZX, 2205.01692, 2208.13790]
Family tree [ZX, Zang, 2309.10849]

[See also Pimentel, Wang, 2205.00013, Qin, ZX, 2208.13790, 2301.07047, Jazayeri, Renaux-Petel, 2205.10340, Qin, Renaux-Petel, Tong, Werth, Zhu, 2506.01555, 2511.00152, etc]

Massive family trees: WYSIWYG

[Liu, ZX, 2412.07843]

- In principle, with family trees known, an arbitrary massive tree graph can be obtained by finishing Mellin integrals, which is a mechanical yet tedious procedure
- Also, the results from PMB + FTD involve more layers of summations than expected --- not optimized in transcendental weight

$$\mathcal{G}(\{E\},\{K\}) = \sum_{\mathsf{a}_1,\cdots,\mathsf{a}_V=\pm} \int_{-\infty}^{0} \prod_{i=1}^{V} \left[\mathrm{d}\tau_i \, \mathrm{i}\mathsf{a}_i (-\tau_i)^{p_i} e^{\mathrm{i}\mathsf{a}_i E_i \tau_i} \right] \prod_{\alpha=1}^{I} D^{(\widetilde{\nu}_\alpha)}_{\mathsf{a}_\alpha \mathsf{a}'_\alpha}(K_\alpha; \tau_\alpha, \tau'_\alpha)$$

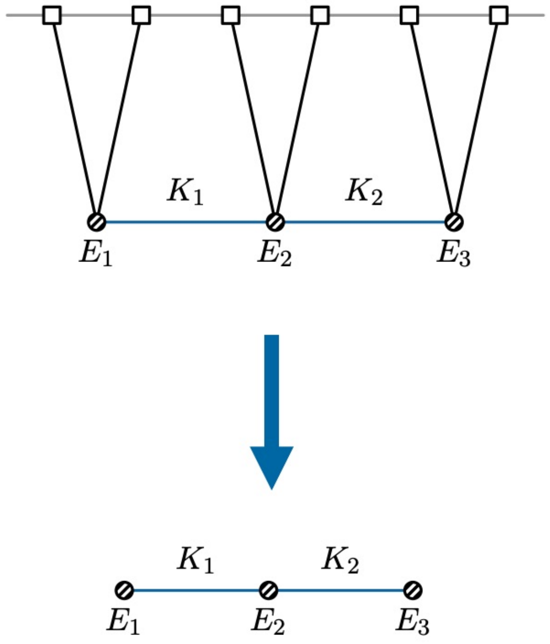
- Question: Can we directly write down the answer for an arbitrary massive tree graph withoutr really computing it?
- Answer: Yes

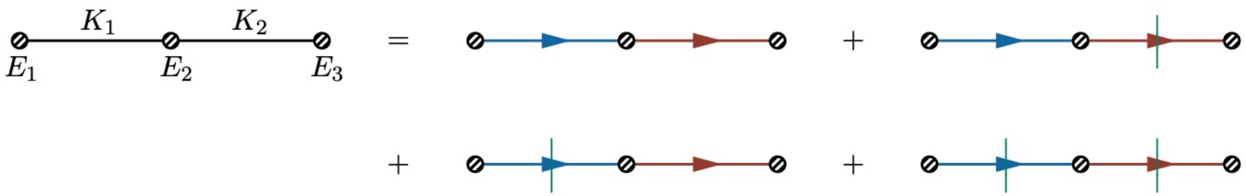
Massive family trees: WYSIWYG

[Liu, ZX, 2412.07843]

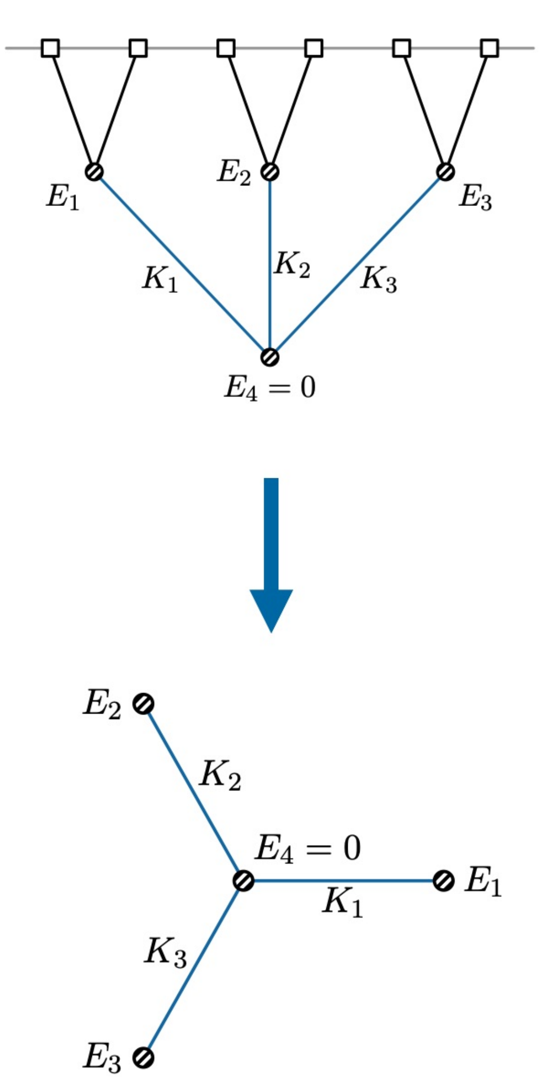
$$\mathcal{G} = \sum_{i \in 2^{\{K\}}} \underset{i}{\mathrm{Cut}}\left[\mathcal{G}\right]$$

- The complete solution to arbitrary massive tree is the sum of the CIS (completely inhom sol) and all of its cuts.
- CIS => massive family tree
- Cuts => “tuned” (♯ or ♭) massive family trees





$$\mathcal{G}_3 = \llbracket 123 \rrbracket + \llbracket 1^{\sharp_1} \rrbracket \Big(\llbracket 2^{\sharp_1} 3 \rrbracket + \llbracket 2^{\flat_1} 3 \rrbracket \Big) + \llbracket 12^{\sharp_2} \rrbracket \Big(\llbracket 3^{\sharp_2} \rrbracket + \llbracket 3^{\flat_2} \rrbracket \Big)$$
$$+ \llbracket 1^{\sharp_1} \rrbracket \Big(\llbracket 2^{\sharp_1 \sharp_2} \rrbracket + \llbracket 2^{\flat_1 \sharp_2} \rrbracket \Big) \Big(\llbracket 3^{\sharp_2} \rrbracket + \llbracket 3^{\flat_2} \rrbracket \Big) + \text{shadows}$$



$$\mathcal{G}_4' = \text{CIS}\left[\mathcal{G}_4'\right] + \sum_{\alpha=1}^{3} \underset{K_\alpha}{\text{Cut}}\left[\mathcal{G}_4'\right] + \sum_{\alpha\neq\beta} \underset{K_\alpha,K_\beta}{\text{Cut}}\left[\mathcal{G}_4'\right] + \underset{K_1,K_2,K_3}{\text{Cut}}\left[\mathcal{G}_4'\right]$$

$$\text{CIS}\left[\mathcal{G}_4'\right] = \llbracket 1\not{4}(2)(3)\rrbracket,$$

$$\sum_{\alpha=1}^{3} \underset{K_\alpha}{\text{Cut}}\left[\mathcal{G}_4'\right] = \llbracket 1^{\sharp_1}\rrbracket \Big(\llbracket 2\not{4}^{\sharp_1}3\rrbracket + \llbracket 2\not{4}^{\flat_1}3\rrbracket\Big) + \llbracket 1\not{4}^{\sharp_2}3\rrbracket \Big(\llbracket 2^{\sharp_2}\rrbracket + \llbracket 2^{\flat_2}\rrbracket\Big)$$
$$+ \llbracket 1\not{4}^{\sharp_3}2\rrbracket \Big(\llbracket 3^{\sharp_3}\rrbracket + \llbracket 3^{\flat_3}\rrbracket\Big) + \text{shadows},$$

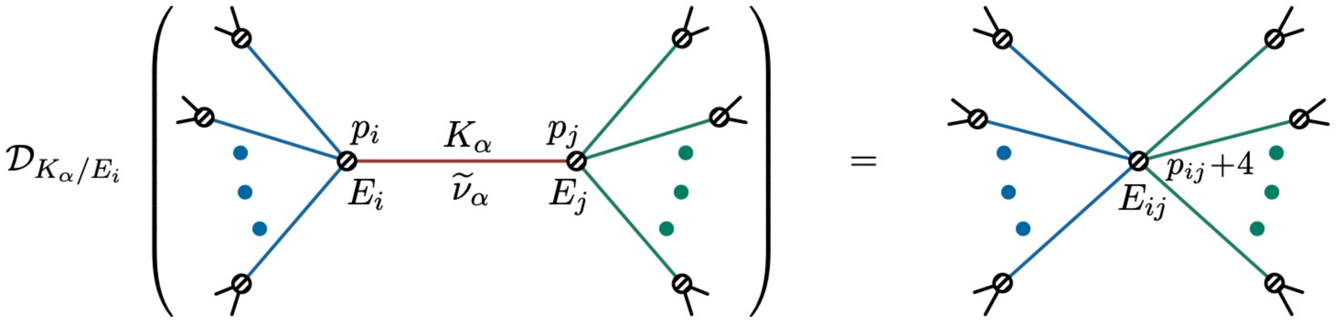
$$\sum_{\alpha\neq\beta} \underset{K_\alpha,K_\beta}{\text{Cut}}\left[\mathcal{G}_4'\right] = \llbracket 1^{\sharp_1}\rrbracket \llbracket 2^{\sharp_2}\rrbracket \Big(\llbracket 3\not{4}^{\sharp_1\sharp_2}\rrbracket + \llbracket 3\not{4}^{\flat_1\sharp_2}\rrbracket + \llbracket 3\not{4}^{\sharp_1\flat_2}\rrbracket + \llbracket 3\not{4}^{\flat_1\flat_2}\rrbracket\Big)$$
$$+ \llbracket 1^{\sharp_1}\rrbracket \Big(\llbracket 2\not{4}^{\sharp_1\sharp_3}\rrbracket + \llbracket 2\not{4}^{\flat_1\sharp_3}\rrbracket\Big)\Big(\llbracket 3^{\sharp_3}\rrbracket + \llbracket 3^{\flat_3}\rrbracket\Big)$$
$$+ \llbracket 1\not{4}^{\sharp_2\sharp_3}\rrbracket \Big(\llbracket 2^{\sharp_2}\rrbracket + \llbracket 2^{\flat_2}\rrbracket\Big)\Big(\llbracket 3^{\sharp_3}\rrbracket + \llbracket 3^{\flat_3}\rrbracket\Big) + \text{shadows},$$

$$\underset{K_1,K_2,K_3}{\text{Cut}}\left[\mathcal{G}_4'\right] = \llbracket 1^{\sharp_1}\rrbracket \llbracket 2^{\sharp_2}\rrbracket \llbracket 3^{\sharp_3}\rrbracket \Big(\llbracket \not{4}^{\sharp_1\sharp_2\sharp_3}\rrbracket + \llbracket \not{4}^{\flat_1\sharp_2\sharp_3}\rrbracket + \llbracket \not{4}^{\sharp_1\flat_2\sharp_3}\rrbracket + \llbracket \not{4}^{\sharp_1\sharp_2\flat_3}\rrbracket$$
$$+ \llbracket \not{4}^{\flat_1\flat_2\sharp_3}\rrbracket + \llbracket \not{4}^{\flat_1\sharp_2\flat_3}\rrbracket + \llbracket \not{4}^{\sharp_1\flat_2\flat_3}\rrbracket + \llbracket \not{4}^{\flat_1\flat_2\flat_3}\rrbracket\Big) + \text{shadows}$$

Differential equations for arbitrary massive trees

[Liu, ZX, 2412.07843]

- An internal line (bulk propagator) is collapsed to 0 or δ by a Klein-Gordon operator
- The KG operator can be pulled out of the integral with IBP at a given vertex
- We obtain a 2nd order diff eq for the graph by picking up a line + one of its two endpoint
- There are a total of 2*I* choices => 2*I* diff eqs for 2*I* indep energy ratios. A complete set!



$$\mathcal{D}_{(\alpha i)}\mathcal{G} = \frac{r_{(\alpha i)}^{p_j+4}\, r_{(\alpha j)}^{p_i+4}}{\big[r_{(\alpha i)} + r_{(\alpha j)}\big]^{p_{ij}+5}} \mathsf{C}_\alpha[\mathcal{G}],$$

$$\mathcal{D}_{(\alpha i)} \equiv \Big(\vartheta_{(\alpha i)} - \frac{3}{2}\Big)^2 + \widetilde{\nu}_\alpha^2 - r_{(\alpha i)}^2\big(\vartheta_{\{i\}} + p_i + 2\big)\big(\vartheta_{\{i\}} + p_i + 1\big)$$

$$r_{(\alpha i)} = \frac{K_\alpha}{E_i} \qquad \vartheta_{(\alpha i)} \equiv r_{(\alpha i)} \frac{\partial}{\partial r_{(\alpha i)}} \qquad \vartheta_{\{i\}} \equiv \sum_{\beta \in \mathcal{N}(i)} \vartheta_{(\beta i)}$$

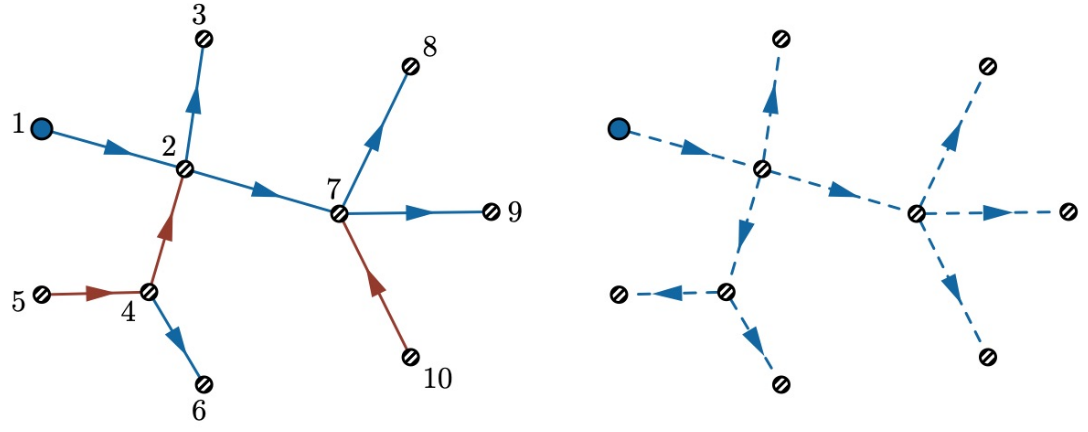
Completely inhomogeneous solution: massive family trees

- Quite remarkably, the CIS has a direct hypergeo rep: $\mathrm{CIS}\left[\mathcal{G}_V\right] = \left[\!\left[\mathscr{P}(1\cdots V)\right]\!\right]$

- The solution expanded in the largest vertex energy ($1/E_1$), indep of the order of other energies

- Picking up a largest energy automatically generates a partial order: massive family tree
  q: a "family parameter" encoding the tree structure:

$$q_i \equiv \widetilde{\ell}_i + 2\widetilde{m}_i + \widetilde{p}_i + 4N_i$$

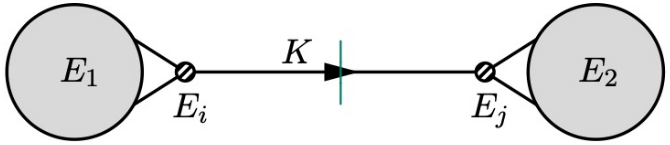


$$\left[\!\left[1\cdots V\right]\!\right] = \sum_{\{\ell,m\}} 2^V \cos(\pi p_{1\ldots V}/2)\Gamma(q_1+p_1+1)$$

$$\times \prod_{i=2}^{V} \frac{(-1)^{\ell_i}}{\ell_i!\left(\frac{\ell_i+q_i+p_i}{2}+\frac{5}{4}\pm\frac{\mathrm{i}\widetilde{\nu}_i}{2}\right)_{m_i+1}} \left(\frac{K_i}{2E_1}\right)^{2m_i+3}\left(\frac{E_i}{E_1}\right)^{\ell_i+p_i+1}$$

Homogeneous solutions: cuts of massive family trees

- The homogeneous solutions are obtained by executing appropriate cuts:



$$\underset{K_\alpha}{\mathrm{Cut}}\left[\widetilde{\mathcal{G}}_V\right] = [\![\widehat{1}\cdots i^\sharp \cdots V_1]\!]\Big\{[\![(V_1+1)\cdots j^\sharp\cdots V]\!] + [\![(V_1+1)\cdots j^\flat\cdots V]\!]\Big\} + \text{ c.c.}$$

- The cut involves certains dressings of massive family trees: augmentation and flattening:

$$[\![\cdots i^\sharp\cdots]\!] \equiv \sum_{m=0}^{\infty}\sqrt{\frac{2}{\pi}}\frac{\Gamma(-m-\mathrm{i}\widetilde{\nu}_\alpha)}{m!}\left(\frac{K_\alpha}{2E_i}\right)^{2m+\mathrm{i}\widetilde{\nu}_\alpha+3/2}[\![\cdots i\cdots]\!]_{p_i\to p_i+2m+\mathrm{i}\widetilde{\nu}_\alpha+3/2}\,,$$

$$[\![\cdots i^\flat\cdots]\!] \equiv \sum_{m=0}^{\infty}\sqrt{\frac{2}{\pi}}\frac{\Gamma(-m+\mathrm{i}\widetilde{\nu}_\alpha)}{m!}\left(\frac{K_\alpha}{2E_i}\right)^{2m-\mathrm{i}\widetilde{\nu}_\alpha+3/2}\left\{\frac{\cos\left[\frac{\pi(p_{\rm tot}+2\mathrm{i}\widetilde{\nu}_\alpha)}{2}\right]}{\cos\left(\frac{\pi p_{\rm tot}}{2}\right)}[\![\cdots i\cdots]\!]\right\}_{p_i\to p_i+2m-\mathrm{i}\widetilde{\nu}_\alpha+3/2}$$

Alternative representation and folded limits

[ZX, Jiaju Zang, 2511.08677]

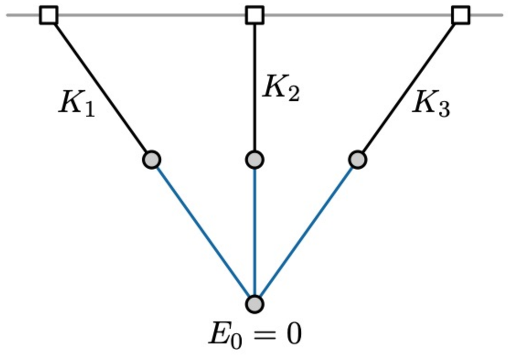
- A change of variables: “dressing” vertex energies: $E_i \to \mathcal{E}_i \equiv E_i + \sum_{\alpha\in\mathcal{N}(i)} \mathsf{s}_{\alpha i} K_\alpha$

  sign label, + or -

- The new energy ratios (u-variables): $u_{\alpha i} \equiv \dfrac{2K_\alpha}{\mathcal{E}_i}$

- New PDE systems and full solutions in terms of u-variables

- The new representation trivializes the folded limit, at which the expressions feature reduced transcendental weight. For instance, the classic triple-massive-exchange graph:



$$\langle \varphi_{\boldsymbol{k}_1}\varphi_{\boldsymbol{k}_2}\varphi_{\boldsymbol{k}_3}\rangle'_{\sigma^3} = \frac{\mu^3\lambda_3 k_{123}^3}{512 k_1^3 k_2^3 k_3^3} \sum_{\mathsf{A}_1,\mathsf{A}_2,\mathsf{A}_3=0,\pm1} 2\cos[\tfrac{\pi}{2}\Delta_{123}^{\mathsf{A}}]\mathcal{C}_1^{\mathsf{A}_1}\mathcal{C}_2^{\mathsf{A}_2}\mathcal{C}_3^{\mathsf{A}_3} u_1^{\Delta_1^{\mathsf{A}}} u_2^{\Delta_2^{\mathsf{A}}} u_3^{\Delta_3^{\mathsf{A}}}$$

$$\times\ {}^{1+2}\mathcal{F}_{0+2}\left[\begin{matrix}\Delta_{123}^{\mathsf{A}}-3 \\ -\end{matrix}\middle|\begin{matrix}\Delta_1^{\mathsf{A}}-1,1;\Delta_2^{\mathsf{A}}-1,1;\Delta_3^{\mathsf{A}}-1,1 \\ \Lambda_{1+}^{\mathsf{A}},\Lambda_{1-}^{\mathsf{A}};\Lambda_{2+}^{\mathsf{A}},\Lambda_{2-}^{\mathsf{A}};\Lambda_{3+}^{\mathsf{A}},\Lambda_{3-}^{\mathsf{A}}\end{matrix}\middle| u_1,u_2,u_3\right]$$

Where are we now?

- Massive tree graphs: solved; WYSIWYG solutions, in hypergeo series
- Loop level: simple 1-loop graphs (massive bubbles) computed, also in hypergeo series
- Analytical structures largely known for all trees and many loops: only poles / branch points of finite degrees
- Conjecture: Any graphic contribution to a renormalized massive cosmological correlator is a multivariate hypergeometric function with only power-law singularities (finite-deg poles or branch points)
- Most of these hypergeo functions are not yet named, and are like “black boxes”
- Then what does the analytical calculation mean other than giving correlators names?
- Why pFq / Appell / Lauricella look like black boxes to us, but sine and cosine do not?

What is analytical computation?

- Using series solutions to define, identify, and represent family trees (hypergeo functions)
- Using the flexibility of FTD to link different reps of family trees => Analytical continuation!

$$[12] = [12] \qquad \frac{1}{\omega_1^{q_{12}}}\ {}_2\mathcal{F}_1\left[\begin{matrix} q_2, q_{12} \\ q_2+1 \end{matrix}\middle| -\frac{\omega_2}{\omega_1}\right] = \frac{\Gamma[q_2]}{\omega_{12}^{q_{12}}}\ {}_2\mathcal{F}_1\left[\begin{matrix} 1, q_{12} \\ q_2+1 \end{matrix}\middle| \frac{\omega_2}{\omega_{12}}\right]$$

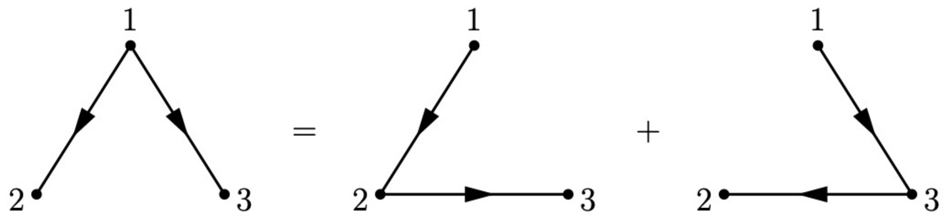
$$[12] + [21] = [1]\,[2] \qquad \frac{1}{\omega_1^{q_{12}}}\ {}_2\mathcal{F}_1\left[\begin{matrix} q_2, q_{12} \\ q_2+1 \end{matrix}\middle| -\frac{\omega_2}{\omega_1}\right] + \frac{1}{\omega_2^{q_{12}}}\ {}_2\mathcal{F}_1\left[\begin{matrix} q_1, q_{12} \\ q_1+1 \end{matrix}\middle| -\frac{\omega_1}{\omega_2}\right] = \frac{\Gamma[q_1, q_2]}{\omega_1^{q_1}\omega_2^{q_2}}$$

$$[123] + [2(1)(3)] = [1]\,[23] \qquad \frac{1}{\omega_1^{q_{123}}}\ {}_{2+1}\mathcal{F}_{1+1}\left[\begin{matrix} q_{123}, q_{23} \\ q_{23}+1 \end{matrix}\middle| \begin{matrix} -, q_3 \\ -, q_3+1 \end{matrix}\middle| -\frac{\omega_2}{\omega_1}, -\frac{\omega_3}{\omega_1}\right] + \frac{1}{\omega_2^{q_{123}}}\mathcal{F}_2\left[q_{123}\middle| \begin{matrix} q_1, q_3 \\ q_1+1, q_3+1 \end{matrix}\middle| -\frac{\omega_1}{\omega_2}, -\frac{\omega_3}{\omega_2}\right]$$

$$= \frac{\Gamma[q_1]}{\omega_1^{q_1}\omega_2^{q_{23}}}\ {}_2\mathcal{F}_1\left[\begin{matrix} q_3, q_{32} \\ q_3+1 \end{matrix}\middle| -\frac{\omega_3}{\omega_2}\right]$$

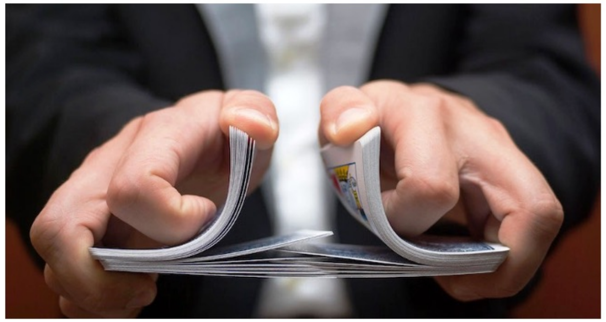
Family trees are further decomposible into chains [Fan, ZX, 2403.07050; 2509.02684]

$$\theta_{21}\theta_{31}(\theta_{32}+\theta_{23}) = \theta_{32}\theta_{21} + \theta_{23}\theta_{31}$$



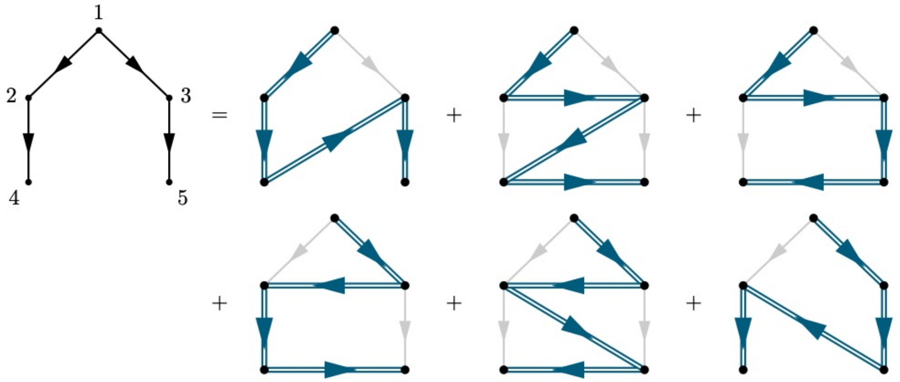
Shuffle product:

$$\begin{aligned} ab \sqcup\!\sqcup\, cd &= abcd + acbd + acdb \\ &\quad + cabd + cadb + cdab \end{aligned}$$



Practically: taking shuffle product recursively among all subfamilies

$$\begin{aligned} [1(24)(35)] &= \{1(24) \sqcup\!\sqcup\, (35)\} \\ &= \{12435\} + \{12345\} + \{12354\} \\ &\quad + \{13245\} + \{13254\} + \{13524\} \end{aligned}$$

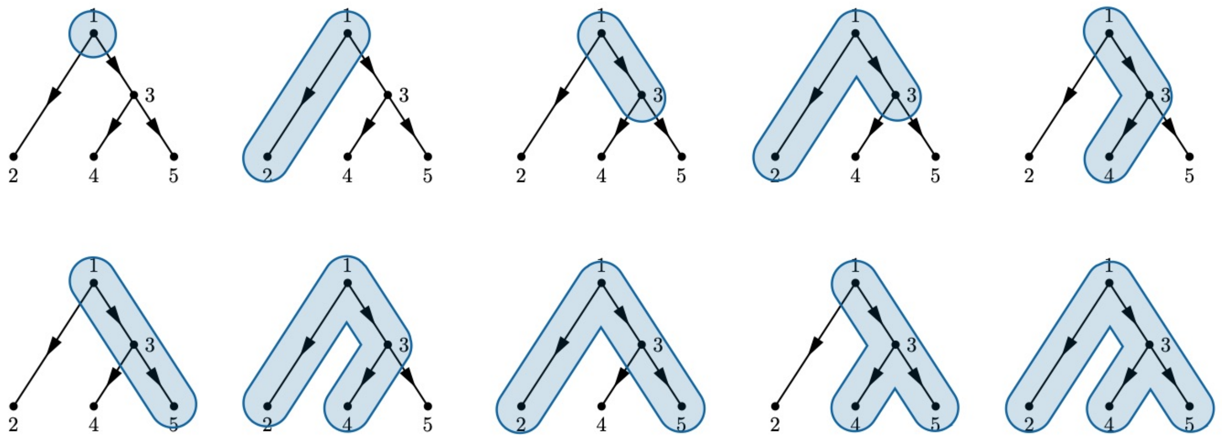


Family chain: standard iterated integrals; Hopf algebra; transcendental weight; Higher weight functions cannot be fully reduced to lower weight functions

Anatomy of family trees

[Fan, ZX, 2509.02684]

A systematic program: charting the singularity map for all family trees



Theorem: *all possible singularities of a family tree come from root-bearing partial energies going to zero or infinity* [proved by Landau analysis]

With appropriate MB reps, we have derived hypergeometric series represenations of an arbitrary family tree around all of its singularities

Example: series in a large partial-energy limit:

$$[\mathscr{P}(\widehat{1}2\cdots N)] = \frac{(-\mathrm{i})^N}{(\mathrm{i}\Omega_1)^{\widetilde{q}_1}} \sum_{n_2,\cdots,n_N=0}^{\infty} \Gamma(\widetilde{q}_1+\widehat{n}_1) \prod_{j=2}^{M} \left[\frac{(\Omega_j/\Omega_1)^{n_j}}{(\widetilde{q}_j+\widehat{n}_j)_{n_j+1}}\right] \prod_{k=M+1}^{N} \left[\frac{(-\omega_k/\Omega_1)^{n_k}}{(\widetilde{q}_k+\widetilde{n}_k)n_k!}\right] .$$

Analytical program: final thoughts and outlooks

- Connection to field theory: Many techniques and and explicit results; They provide useful tools and explicit data for answering more challenging theoretical questions, e.g., nonperturbative properties of correlators and their relations to fundamental principles?

- Connection to maths: We have to deal with new hypergeometric functions that display intriguing structures: family trees; singularity structures; connecting formulae; cutting and dispersion; relation to polytopes

- Connection to data: Numerical strategies! Series reps; analytical continuation; approximations?
  --- All computations must be initiated analytically and finished numerically, the only question being where to execute the analytical-to-numerical transition

- We hope that some of the analytical progress can provide new insights and better answers!

Thank you!