Momentum Space for Cosmological Correlators

Arthur Poisson

work with N. Belrhali, S. Renaux-Petel & D. Werth

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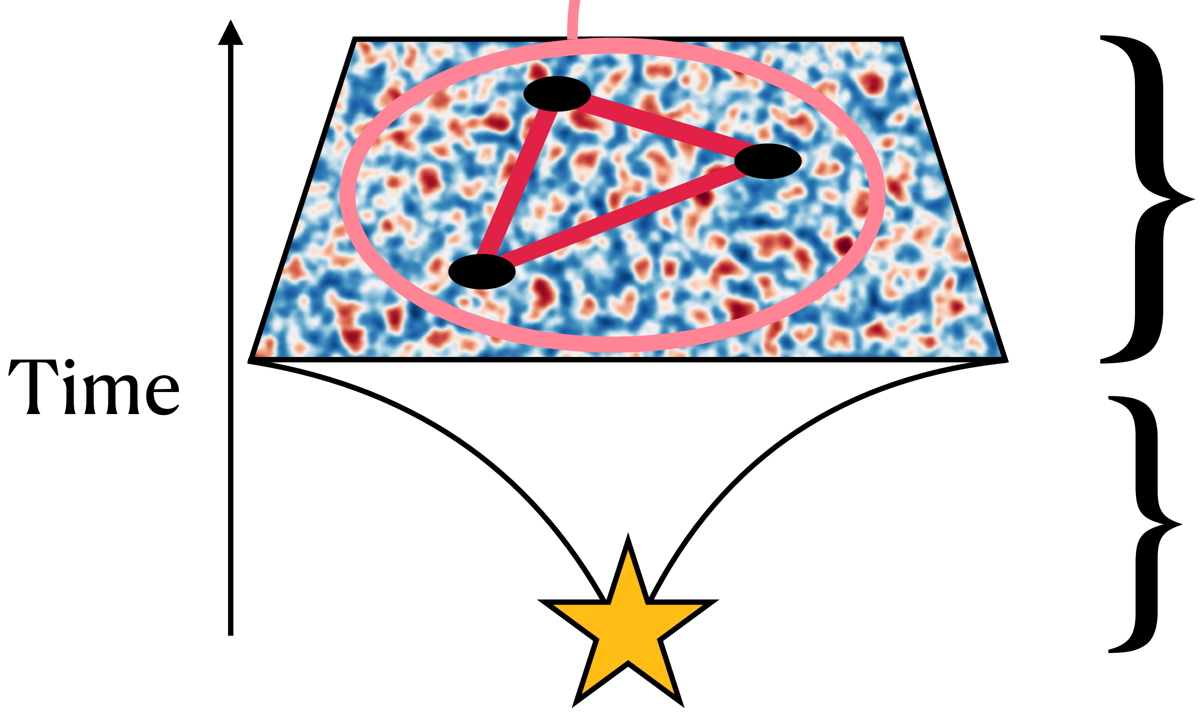
Cosmological Correlators

- **Inflationary observables**: correlation functions at the final surface of inflation.

$$\langle \Omega | \mathcal{O}_{k_1}(\tau_f) \ldots \mathcal{O}_{k_n}(\tau_f) | \Omega \rangle$$



- Inflation can be **approximated** by a de-Sitter space time.

$$\varepsilon = -\frac{\dot{H}}{H^2} \ll 1$$

Lack of a Proper Momentum Space

- In QFT, $\langle \Omega | \mathcal{O}_{k_1}(\tau_f) .. \mathcal{O}_{k_n}(\tau_f) | \Omega \rangle \sim \underset{\text{Space-time}}{\int} \left(\begin{array}{c} \text{Feynman} \\ \text{diagrams} \end{array} \right)$

- In flat space-time translation invariance:

Great simplifications in Space and time Fourier space:

$$\mathcal{O}(t, \mathbf{x}) \sim e^{-ik\cdot x + i\omega t} \implies \langle \Omega | \mathcal{O}_{k_1}(\tau_f) .. \mathcal{O}_{k_n}(\tau_f) | \Omega \rangle \sim \delta^{d+1}(\sum k_i^\mu) G(\{k_i\})$$

- In Inflation: only spatial translation invariance

One can only go to spatial Fourier space:

$$\mathcal{O}(t, \mathbf{x}) \sim e^{-ik\cdot x} u_k(\tau) \implies \langle \Omega | \mathcal{O}_{k_1}(\tau_f) .. \mathcal{O}_{k_n}(\tau_f) | \Omega \rangle \sim \delta^{d}(\sum k_i) \underset{\text{time}}{\int} \left(u_k(\tau)'s \right)$$

Is it possible to find a proper momentum space in dS?

Outline

- de-Sitter geometry and symmetries.
  - de-Sitter: generalities
  - Building the Hilbert space using the symmetries

- New momentum space for de-Sitter: Kontorovich-Lebedev-Fourier space.
  - New representation for fields in dS
  - Application: spectral representation for propagators.

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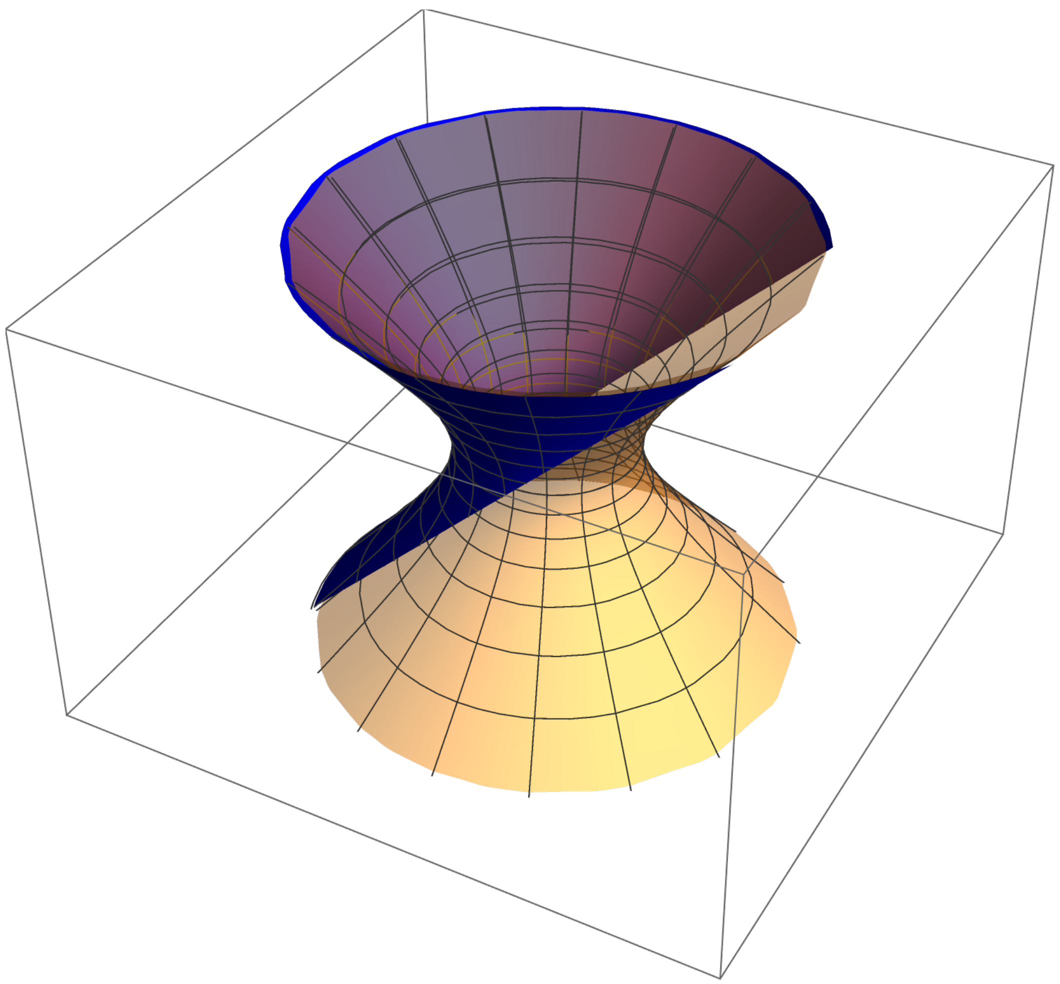
de-Sitter Space-Time

- dS_{d+1} is defined as an hyperboloid surface in $\mathbb{M}^{d+1,1}$:



$$X^A X^B \eta_{AB} = + H^{-2}$$

- FLRW slicing: Poincaré Patch.

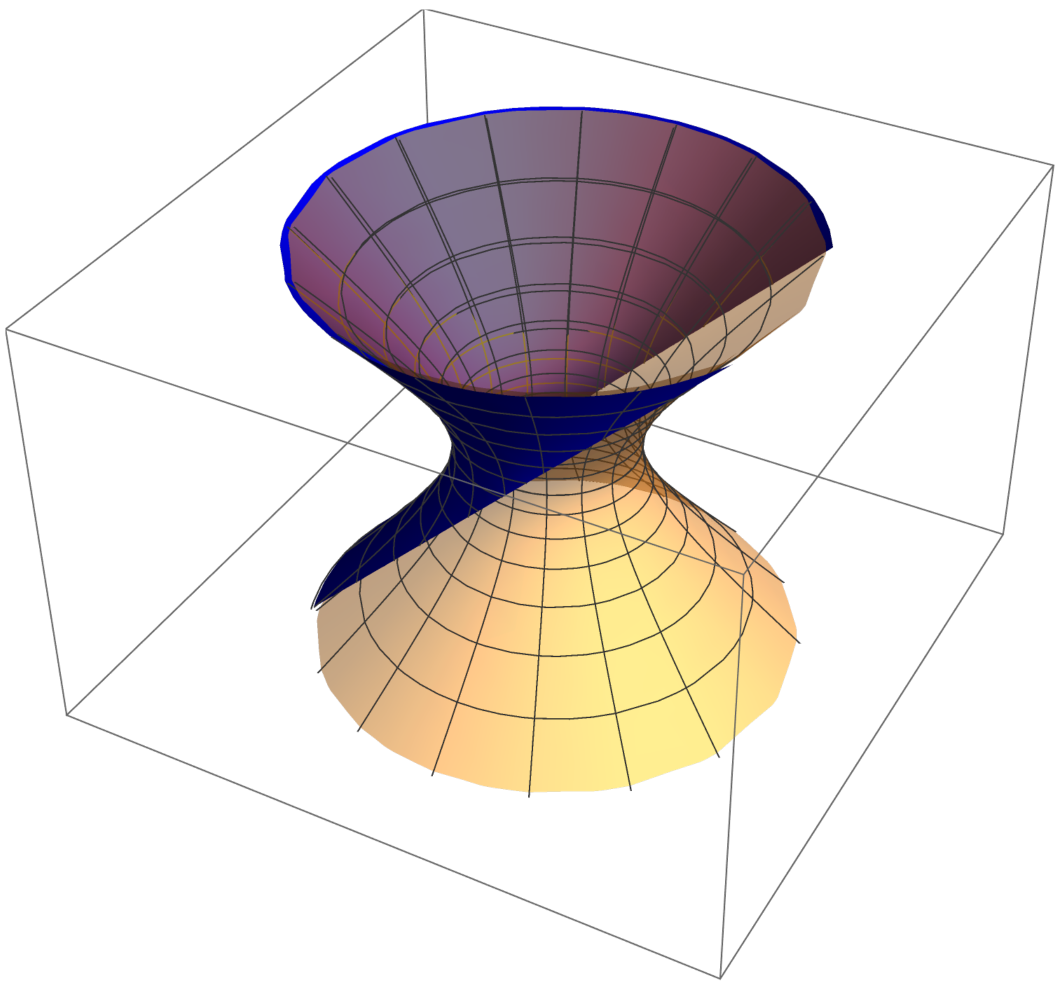
$$ds^2 = \frac{-d\tau^2 + d\mathbf{x}^2}{H^2\tau^2} = -dt^2 + e^{Ht}d\mathbf{x}^2$$

Conformal time	Cosmic time
$\tau \in]-\infty, 0]$	$t \in]-\infty, +\infty[$

- Future boundary: plane at $\tau \to 0$.
- The Poincaré Patch only covers half of de-Sitter Space time.

A Maximally Symmetric Space

- dS_{d+1} is defined as an hyperboloid surface in $\mathbb{M}^{d+1,1}$:



$$X^A X^B \eta_{AB} = +H^{-2}$$

- The definition is Lorentz invariant: $\mathrm{SO}(d+1,1)$ symmetry.

- Generators: $$J_{AB} = i\left(X_A \partial_B - X_B \partial_A\right)$$

- Obey the Lorentz Lie algebra:

$$[J_{AB}, J_{CD}] = i\left(\eta_{BC} J_{AD} - \eta_{AC} J_{BD} + \eta_{AD} J_{BC} - \eta_{BD} J_{AC}\right)$$

Representation Theory

- QFT in de-Sitter ~ Hilbert space $\mathscr{H}$

Labels unitary irreducible representations of $\mathrm{SO}(d+1,1)$

$$\mathscr{H} = \bigoplus_{\mu} \mathcal{V}_\mu$$

Invariant subspace of μ representation

- Casimir operator C: $[C, J_{AB}] = 0 \;\; \forall \;\; J_{AB}$

- **Schur Lemma**: Casimir operator labels UIRs.

$$C = \bigoplus_{\mu} M_\mu^2 \mathbb{I}_d^\mu \implies C|\mu\rangle = M_\mu^2 |\mu\rangle$$

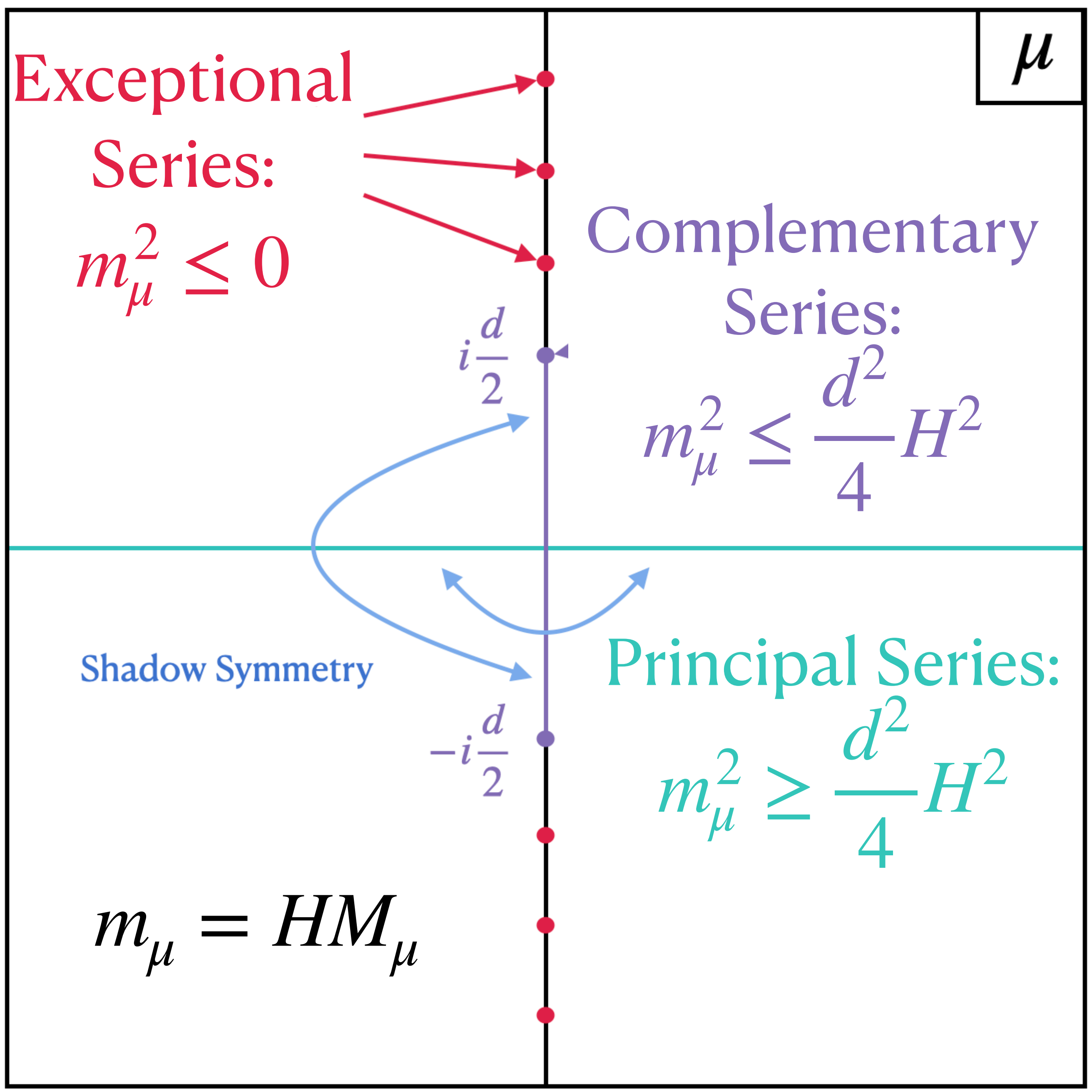
Eigenvalue $M_\mu^2 = \mu^2 + \dfrac{d^2}{4}$

Any state of $\mathscr{V}_\mu$

Representation Theory

- QFT in de-Sitter ~ Hilbert space $\mathscr{H} = \bigoplus_{\mu} \mathcal{V}_{\mu}$

Invariant subspace of μ representation



$$C\,|\mu\rangle = M_\mu^2\,|\mu\rangle$$

Any state of $\mathscr{V}_\mu$

Eigenvalue $M_\mu^2 = \mu^2 + \frac{d^2}{4}$

- C hermitian $\rightarrow M_\mu^2 \in \mathbb{R}$.

Poincaré Momentum Space

- Splitting of the generators: $d-$dimensional conformal algebra.

$$J_{ij} = M_{ij}\,, \qquad J_{0,d+1} = D\,, \qquad J_{d+1,i} = \frac{P_i + K_i}{2}\,, \qquad J_{0,i} = \frac{P_i - K_i}{2}$$

Rotations

Dilatations

Translations

Special Conformal Transformations

- Diagonalize translations: momentum space.

$$P_i \,|\mu, \mathbf{k}\rangle = k_i \,|\mu, \mathbf{k}\rangle$$

- This gives a basis for the eigenspaces $\mathscr{V}_\mu$

$$\mathscr{V}_\mu = \mathrm{Span}_k(|\mu, \mathbf{k}\rangle)$$

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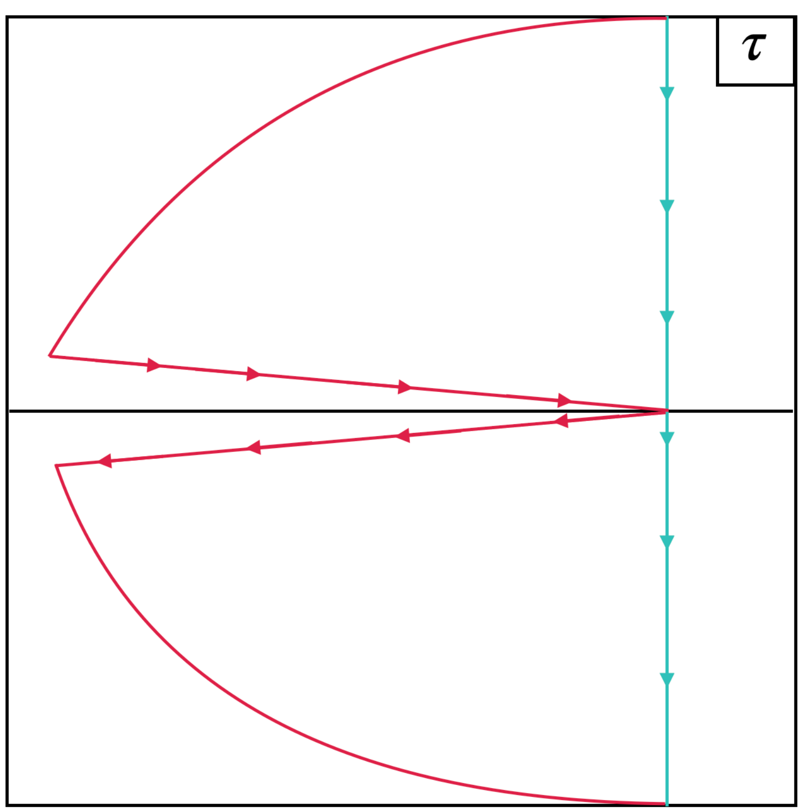
Vacuum Correlation Functions

- We want to compute $\langle\Omega|\mathcal{O}_{k_1}(\tau)\ldots\mathcal{O}_{k_n}(\tau)|\Omega\rangle$

Bunch-Davies
Vacuum state

- In order to properly define $|\Omega\rangle$: $\mathrm{Im}(\tau)\neq 0$:

$$\langle\Omega|\,\mathcal{O}(\boldsymbol{X}^a)\,|\Omega\rangle = \langle 0|\left[\bar{\mathrm{T}}\,e^{i\int_{-\infty+}^{t} H_{\mathrm{I}}(t')\mathrm{d}t'}\right]\mathcal{O}(X^a)\left[\mathrm{T}\,e^{-i\int_{-\infty-}^{t} H_{\mathrm{I}}(t')\mathrm{d}t'}\right]|0\rangle$$



- Each branch can be rotated to the imaginary axis.
- This maps de-Sitter to Euclidean Anti-de-Sitter:

$$ds^2 = \frac{dz^2 + d\mathbf{x}^2}{H^2 z^2}\,, \quad z = -i\tau$$

Position Space in EAdS

- We define the position basis:

$$|z,\boldsymbol{x}\rangle = \int_{\text{KLF}} \mathrm{d}\mu \mathcal{N}_{(\mu)} \frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d} \left[\Phi_{\boldsymbol{k}}^{(\mu)}(z,\boldsymbol{x})\right]^* |\mu,\boldsymbol{k}\rangle \quad \text{Where} \quad \mathcal{N}_{(\mu)} = \frac{\mu}{\pi}\sinh(\pi\mu)$$

μ only runs along the principal series

de-Sitter density of states

- The wave-function obeys the **Euclidean** Casimir equation with solution:

$$\Phi_{\boldsymbol{k}}^{(\mu)}(z,\boldsymbol{x}) = \frac{H^{\frac{d+1}{2}}}{\sqrt{\pi}} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} z^{d/2} K_{i\mu}(kz)$$

- This defines the Kontorovich-Lebedev-Fourier (KLF) transformation for EAdS functions:

$$f(z,\boldsymbol{x}) = \int_{\text{KLF}} \mathrm{d}\mu \mathcal{N}_{(\mu)} \frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d} \Phi_{\boldsymbol{k}}^{(\mu)}(z,\boldsymbol{x}) f_{\boldsymbol{k}}^{(\mu)} \qquad f_{\boldsymbol{k}}^{(\mu)} = \int_{\text{EAdS}} \frac{\mathrm{d}z\mathrm{d}^d\boldsymbol{x}}{(Hz)^{d+1}} \left[\Phi_{\boldsymbol{k}}^{(\mu)}(z,\boldsymbol{x})\right]^* f(z,\boldsymbol{x})$$

And

KLF propagator

- One can rewrite the free field action in KLF space:

$$S_{\text{free}} = \frac{1}{2}\int_{EAdS_{d+1}} \mathrm{d}z\mathrm{d}^d x\sqrt{-g}\left(-(\partial\varphi)^2 - m_\mu^2\varphi^2\right) = \frac{H^2}{2}\int_{\text{KLF}} \frac{\mathrm{d}\nu\mathrm{d}^d k\mathcal{N}_\nu}{(2\pi)^d}\varphi_{\mathbf{k}}^\nu\left(\nu^2-\mu^2\right)\varphi_{-\mathbf{k}}^\nu$$

- The Feynman propagator takes a very simple form:

$$G_\mu(\nu) = \frac{i}{\nu^2-\mu^2}$$

- Flat space analogous:

$$G_m(p) = \frac{i}{m^2+p^2}$$

- Easy way to map KLF correlation to real time.

- This an be written In real dS time as a spectral representation:

$$G^\mu_{++}(\tau_1,\tau_2;k) = +\frac{i}{H^2}\int \mathrm{d}\nu\mathcal{N}_\nu\frac{u_k^*(\tau_1,\nu)u_k^*(\tau_2,\nu)}{\nu^2-\mu^2}$$

S. Melville, G. Pimentel: 2404.05712
D. Werth: 2409.02072

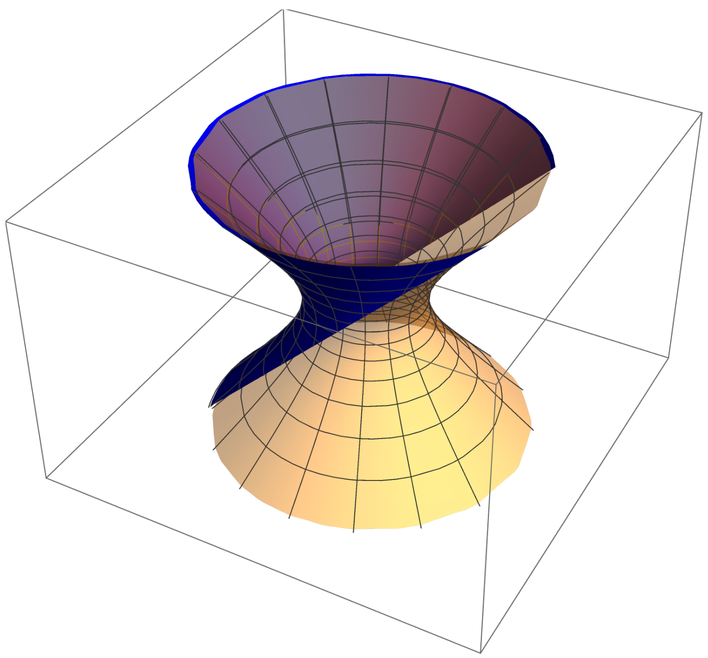
Conclusion

- **Space-time expansion:** inflation not invariant under time translations.

$\Longrightarrow$ No Fourier space for the time!

$$\varphi(x,t) = e^{-ikx} \underbrace{u_k(t)}_{\neq e^{i\omega t}}$$

- Inflation $\simeq$ de-Sitter space-time = maximally symmetric.



$\Longrightarrow$ Using these symmetries, we find a new momentum space

- KLF space: $f(z,\boldsymbol{x}) = \int_{\mathrm{KLF}} \mathrm{d}\mu \mathcal{N}_{(\mu)} \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} \Phi_{\boldsymbol{k}}^{(\mu)}(z,\boldsymbol{x}) f_{\boldsymbol{k}}^{(\mu)}$

- New language for QFT in dS space-time.