創発的研究支援事業
Fusion Oriented REsearch for disruptive Science and Technology

Advances in the separate universe approach: from theory to applications

Yuko Urakawa (KEK IPNS)

T. Tanaka & Y.U. JCAP **07** (2021) 051,
Phys. Rev. Lett. 132(2024) 23, 231003
JCAP **07** (2025) 045

P. Saha, Y. Tada, & Y.U. in progress

Dec 3 2025 @ Paris

Inflation and gradient expansion

To connect prediction of inflation with observations,
we need to solve large scale evolution (k/aH << 1).

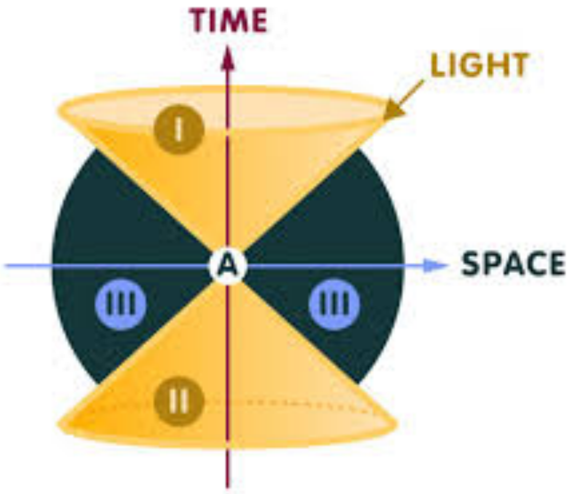
Useful method for ζ

Gradient expansion (GE) → delta N formalism

Salopek&Bond (90)
Shibata & Sasaki (90),
Deruelle & langlois (94),...

Starobinsky (82, 85),
Sasaki & Stewart (95),
Sasaki & Tanaka (98),
Lyth, Malik, & Sasaki (04),
... Lyth &Rodriguez (05)....

separate universe approach, based on causality



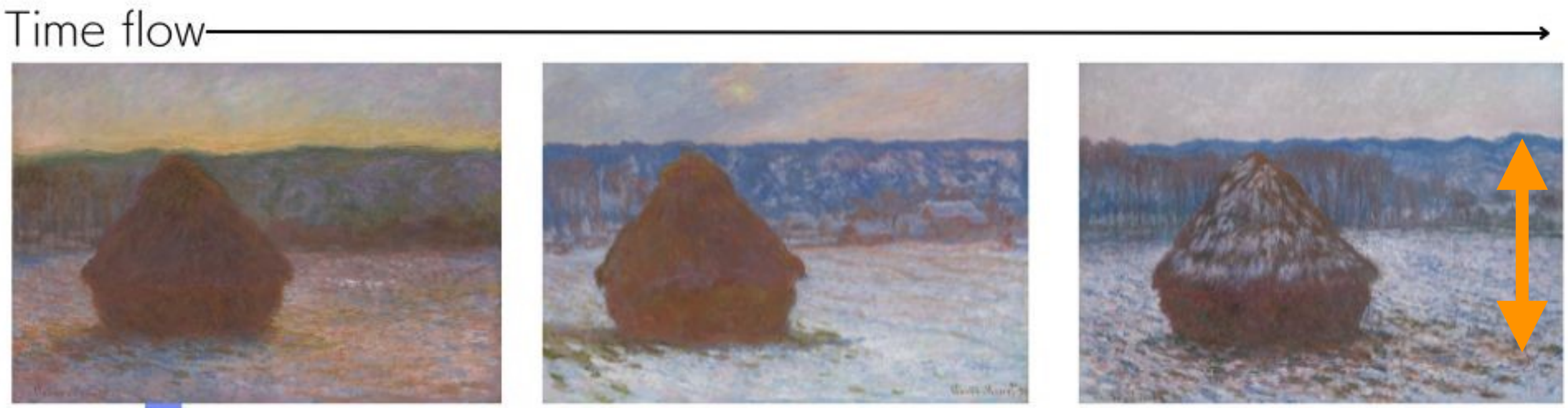
Separate Universe approach ~ Mosaic art

Salopek&Bond(90)

Actual Landscape (Evolution equation which depends on time and space)

Separate Universe approach ~ Mosaic art

Salopek&Bond(90)



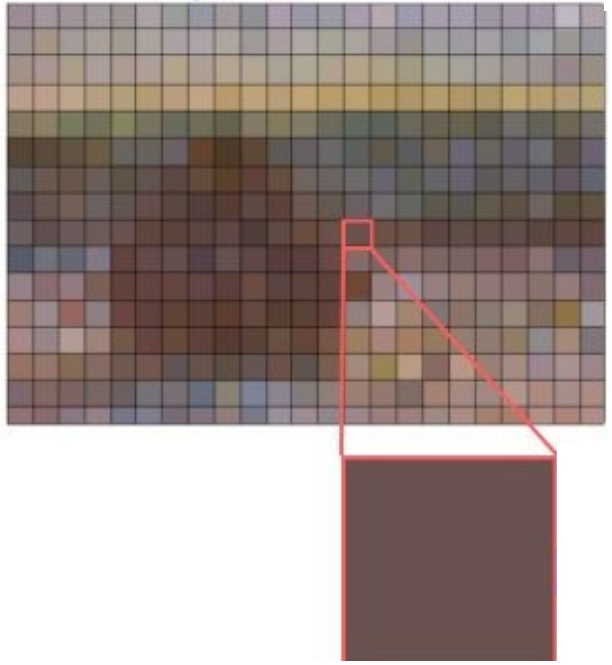
scale of interest

Actual Landscape (Evolution equation which depends on time and space)

coarse-graining w/causal region

Mosaic art

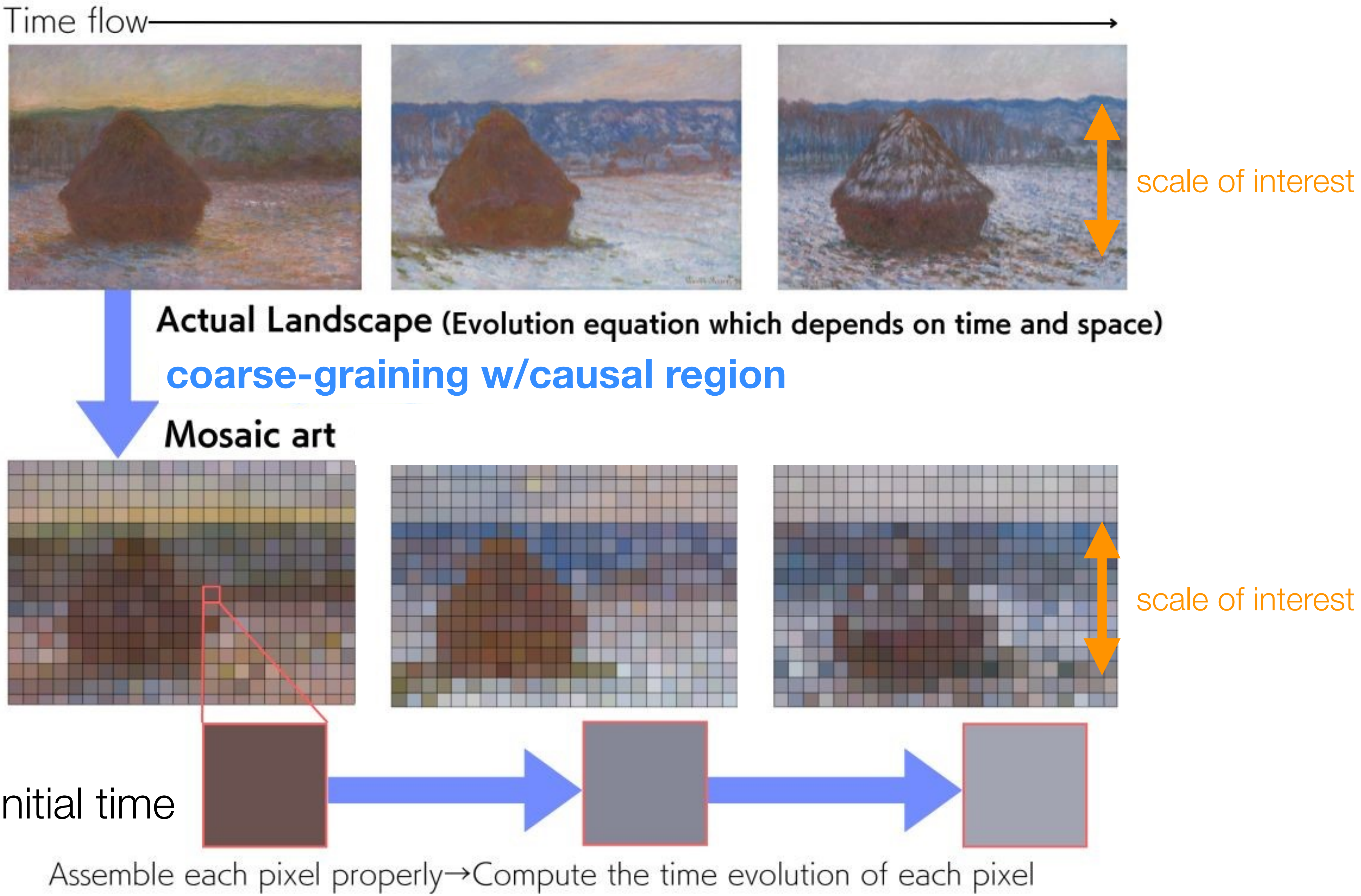
Different region evolves independently.



at initial time

Separate Universe approach ~ Mosaic art

Salopek&Bond(90)



Time flow

scale of interest

Actual Landscape (Evolution equation which depends on time and space)

coarse-graining w/causal region

Mosaic art

scale of interest

at initial time

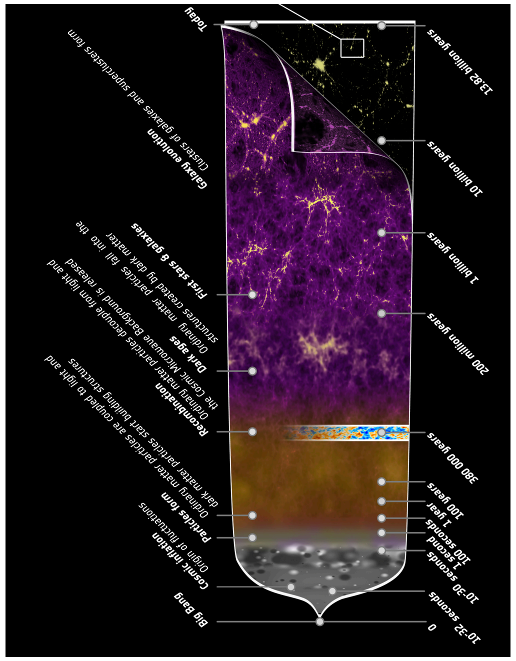
Assemble each pixel properly→Compute the time evolution of each pixel

delta N formalism

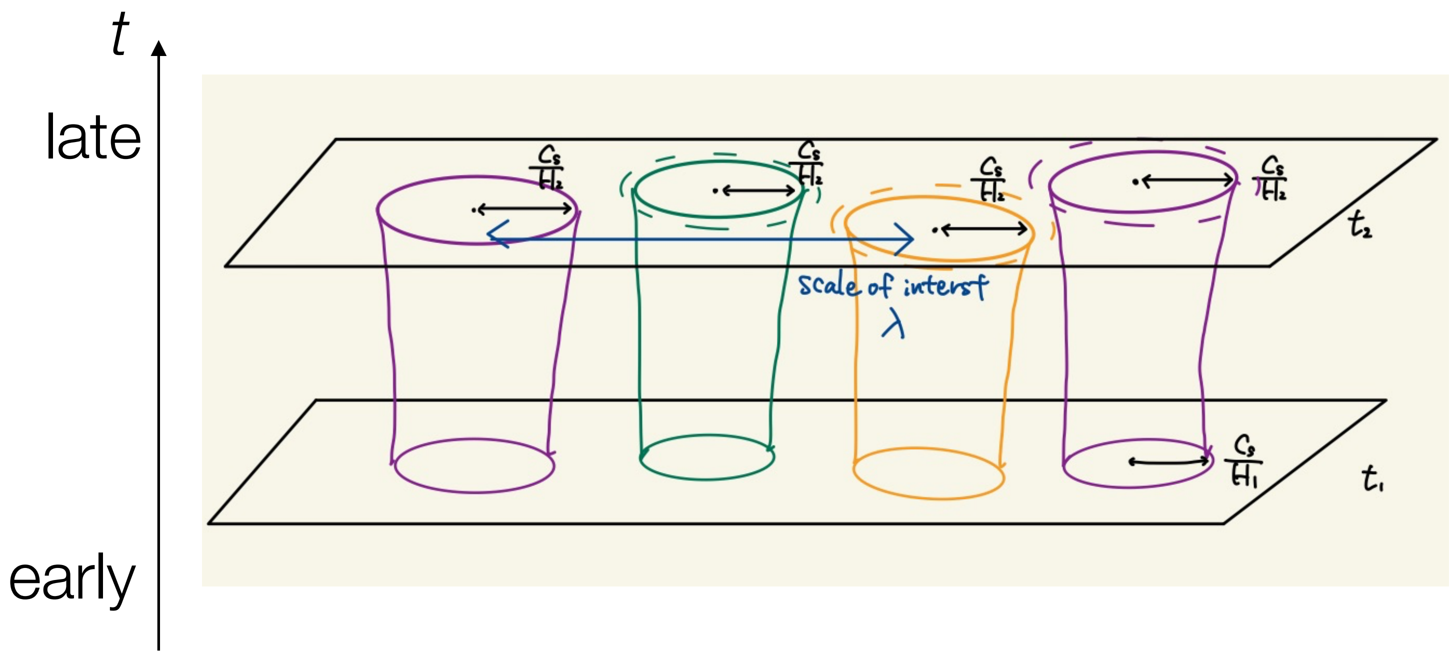
Starobinsky (82, 85),
Sasaki & Stewart (95),
Sasaki & Tanaka (98),

Evolution of inhomogeneous Universe
= Evolution of glued numerous homogeneous universes

Scale of interest λ >> Smoothing scale λ_S ~ Size of casual patch ~ c/H



= (☆)



Fine-grained view

Solving PDEs

Coarse-grained view

Solving ODEs (Inhomogeneity: Different ICs)

Geometrical (→model indep.&Non-perturbative) computation

$\zeta(t, \mathbf{x})$ ← Compute the cosmic expansion $a(t_2)/a(t_1)$(=exp[N]) of each patch

Starobinsky (82, 85),
Sasaki & Stewart (95),
Sasaki & Tanaka (98),
Lyth, Malik, & Sasaki (04),...

δN formalism

$(☆) \longrightarrow$ δN formalism

$$\lim_{\epsilon\to 0}\varphi^a(t,\,\boldsymbol{x}) = \bar\varphi^a(t;\{\bar\varphi^{a'}(t_*) = \varphi^{a'}(t_*,\,\boldsymbol{x})\}')$$

\+

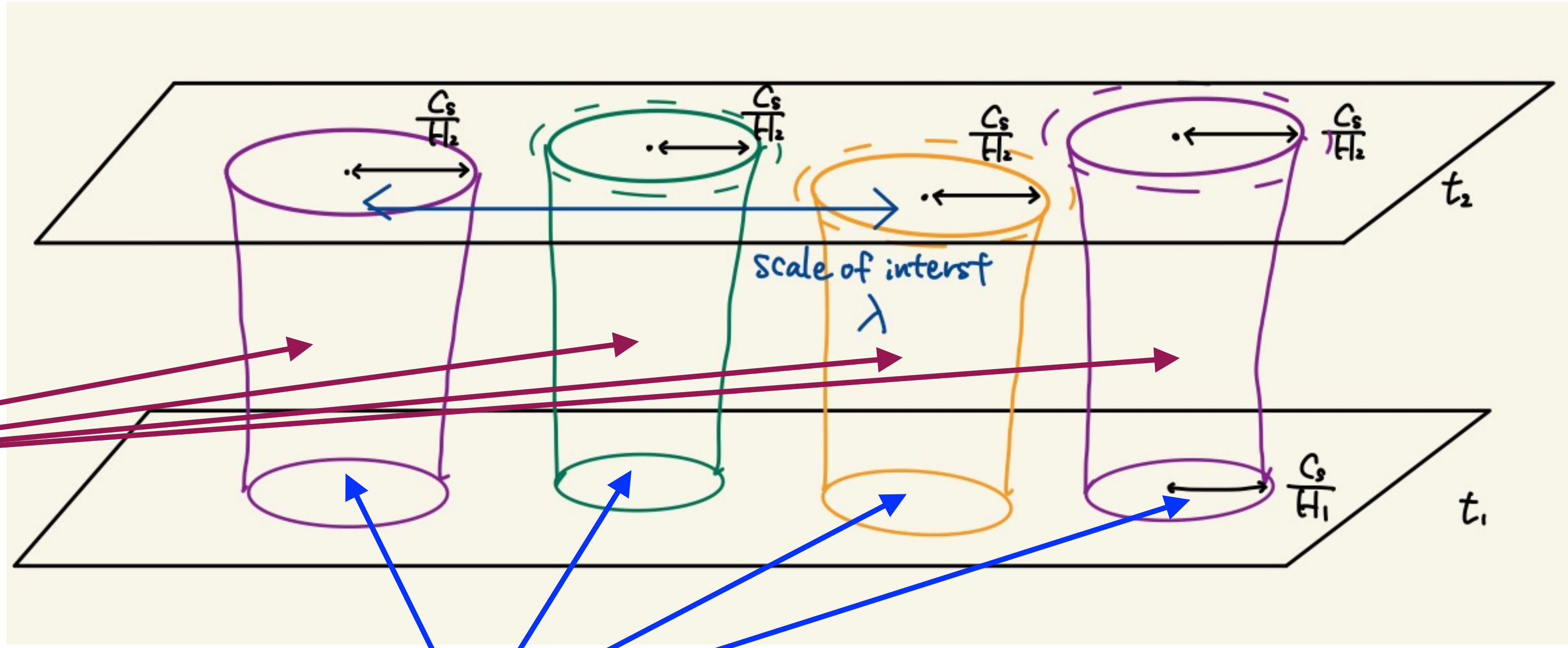
Geometrical relation

$$\psi(t,\,\mathbf{x}) = \frac{1}{3}\int^t dt' N(t',\,\mathbf{x})K(t',\,\mathbf{x})$$

inhomogeneous e-folding number

spatial gauge w/Nᵢ=0

3-dim spatial metric $g_{ij} = e^{2\psi}\,\gamma_{ij}$



reheating surface w/uniform density

Hubble crossing w/flat

homogeneous e-folding

IC: Determined by QFT

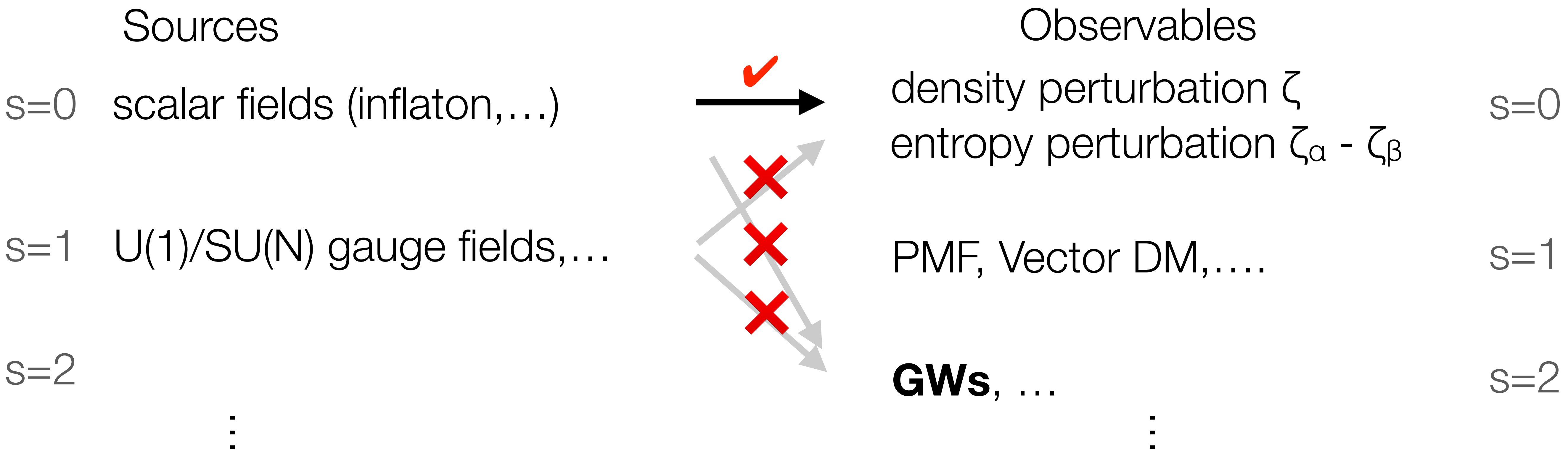
Pros and Cons of δN

Pros

- Drastic simplification down to back-of-envelope computation
  → Widely used to solve non-linear evolution
- Useful to develop intuitive understanding on generation mechanism of ζ

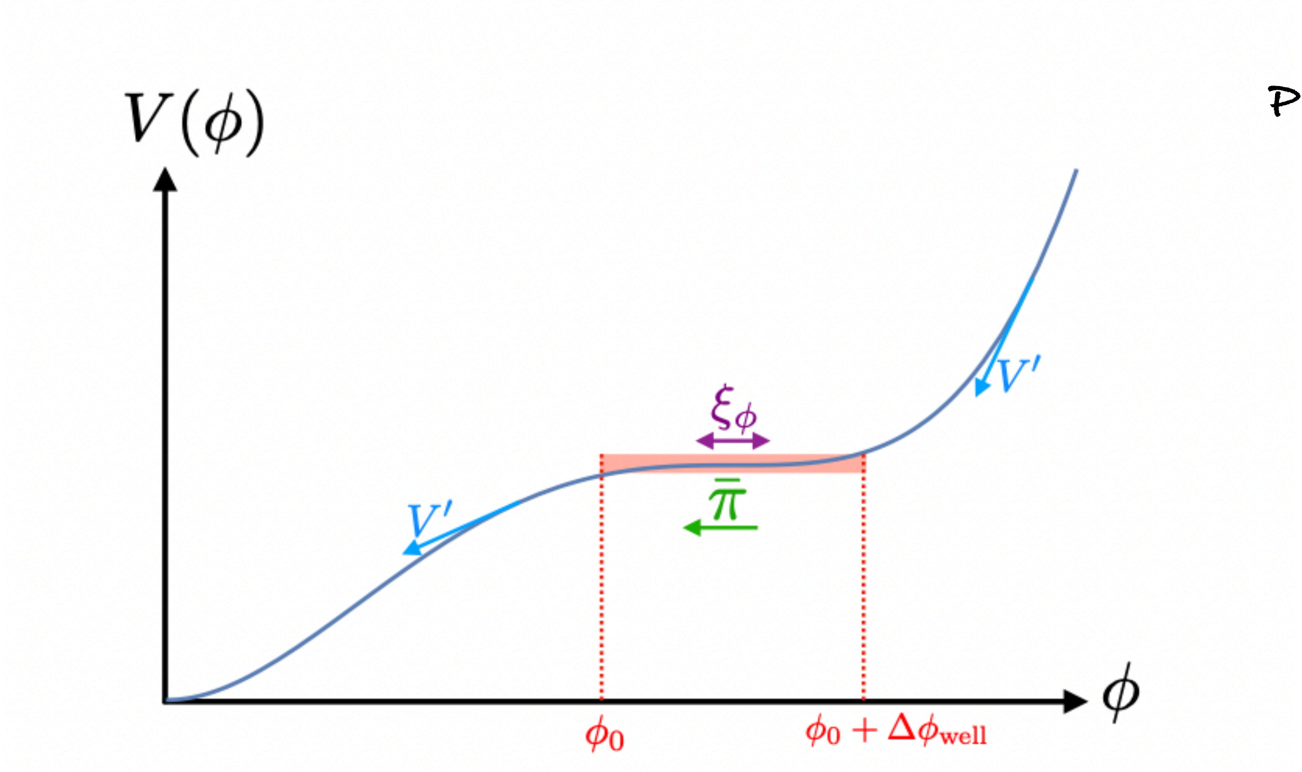
Cons

Limited applicability



	Sources	Observables	
s=0	scalar fields (inflaton,…)	density perturbation ζ entropy perturbation ζ_α - ζ_β	s=0
s=1	U(1)/SU(N) gauge fields,…	PMF, Vector DM,….	s=1
s=2		**GWs**, …	s=2
	⋮	⋮	

A modern application 1: Impact of diffusion



Pattison et al. (2017, 2021), Ezquiaga et al. (2020).
Figueroa et al. (2020),….

Figure from Pattison et al. (2017)

- Significant diffusion effect
- Interesting ballpark for PBH formation
- Deviation from Gaussian (Heavy tail)?

Can we characterize the heavy tail via δN or we need “beyond”?
…. if so, when do we need the extension?

A modern application 2: Axionic inflation

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\alpha}{4}\frac{\phi}{f}F_{\mu\nu}\tilde{F}^{\mu\nu} \qquad \tilde{F}^{\mu\nu} = \frac{1}{2\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

axion ϕ may or may not be inflaton

- Monodromy inflation — Characteristic power spectrum — Silverstein &Westphal(08)
- Efficient preheating and GW production — Adshead et al. (2015, 2018, 2023),....
- Magnetogenesis — Garreston, Field, Carroll(92), Finelli and Gruppuso (2001),... Fujita et al. (2015), Adshead et al. (2016), Patel, Tashiro, Y.U. (2019),...
- Large non-Gaussianity — Barnaby and Peloso (2010), Barnaby et al. (2011),...
- Enhancement of primordial perturbations and primordial blackholes

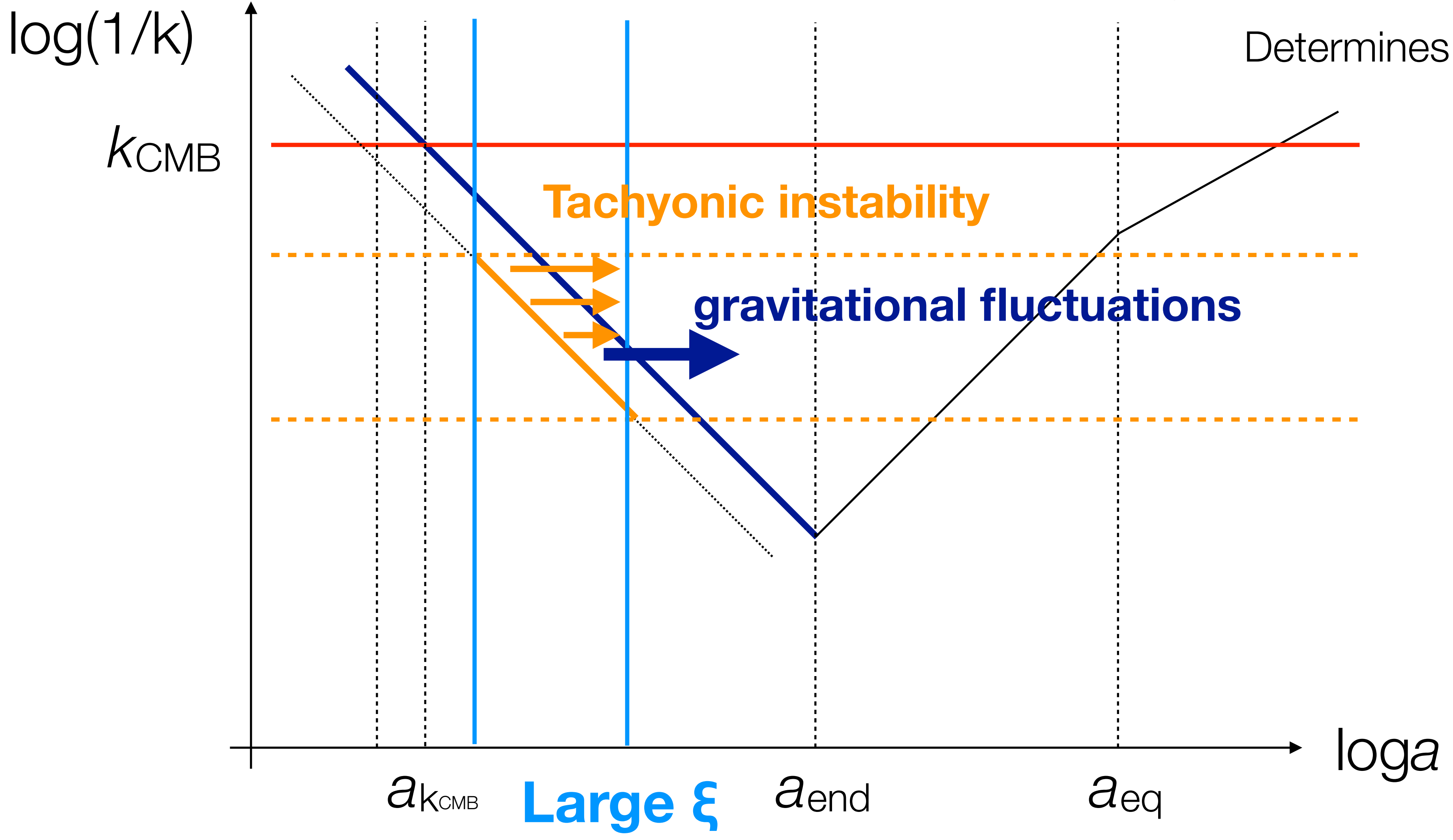
Linde, Mooji, and Pajer (2012),.....

Warm inflation? Baryogenesis? Connection to string theory/swampland? Chiral gravitational waves?…..

A modern application 2: Axionic inflation

k_{CMB}: Scale of interest, e.g., CMB scale

Determines the GE parameter



tachyonic instability for $k < k_h \sim aH\xi$

$$|\xi| = \sqrt{2\varepsilon}\alpha\frac{M_{\mathrm{pl}}}{f}$$

For $k/aH \leqq 1$, impacts of gravitational fluctuations need to be taken into account.

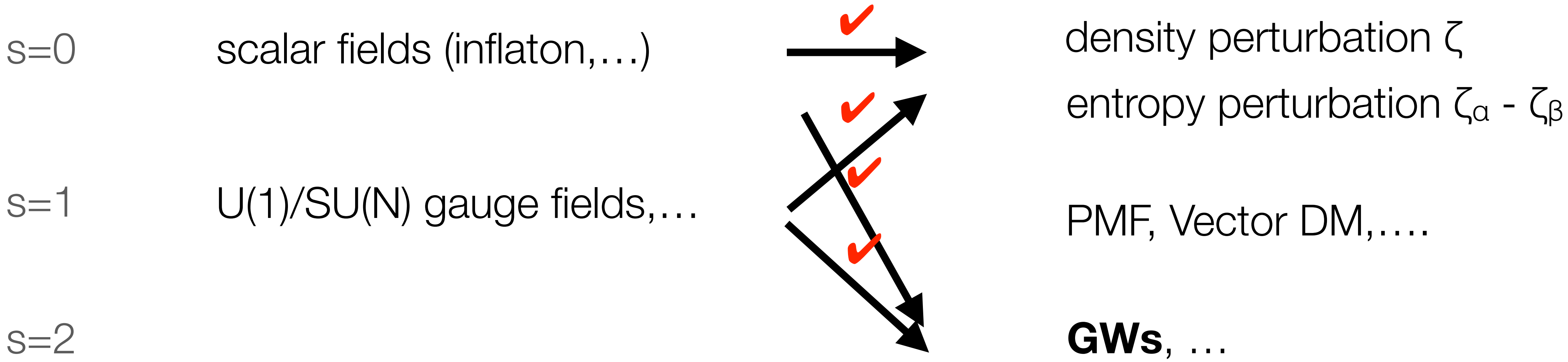
(For preheating, important dynamics happens within each horizon patch.)

Two generalizations

1) Going beyond scalar systems

w/ Tanaka (2021, 2023, 2024)

- From arbitrary (integer) spins to arbitrary spins



s=0 scalar fields (inflaton,…)

density perturbation ζ

entropy perturbation $\zeta_\alpha - \zeta_\beta$

s=1 U(1)/SU(N) gauge fields,…

PMF, Vector DM,….

s=2

GWs, …

- Clarification of their validity

From gradient expansion

2) Going beyond gradient expansion

w/ Saha, Tada (in prep)

Scalar fields system where impacts of subhorizon modes can be potentially important. Incl. initial NG,…

Separate universe

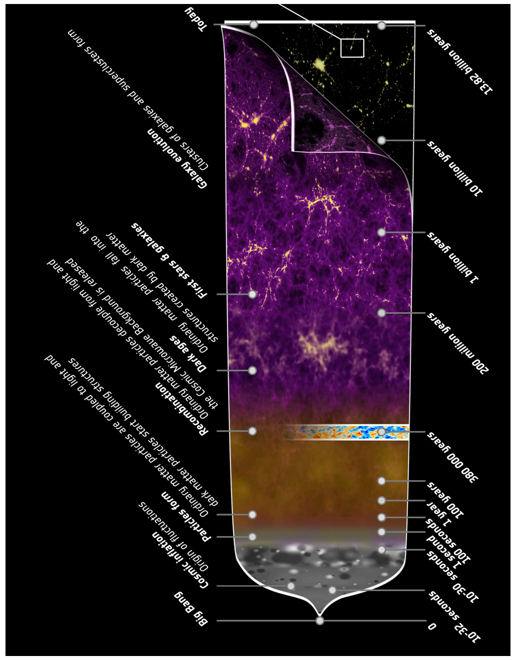
Salopek&Bond(90)

(☆) Evolution of inhomogeneous Universe

= Evolution of glued numerous homogeneous Universes

Scale of interest λ >> Smoothing scale λ_S > Size of causal patch

~ c_S/H



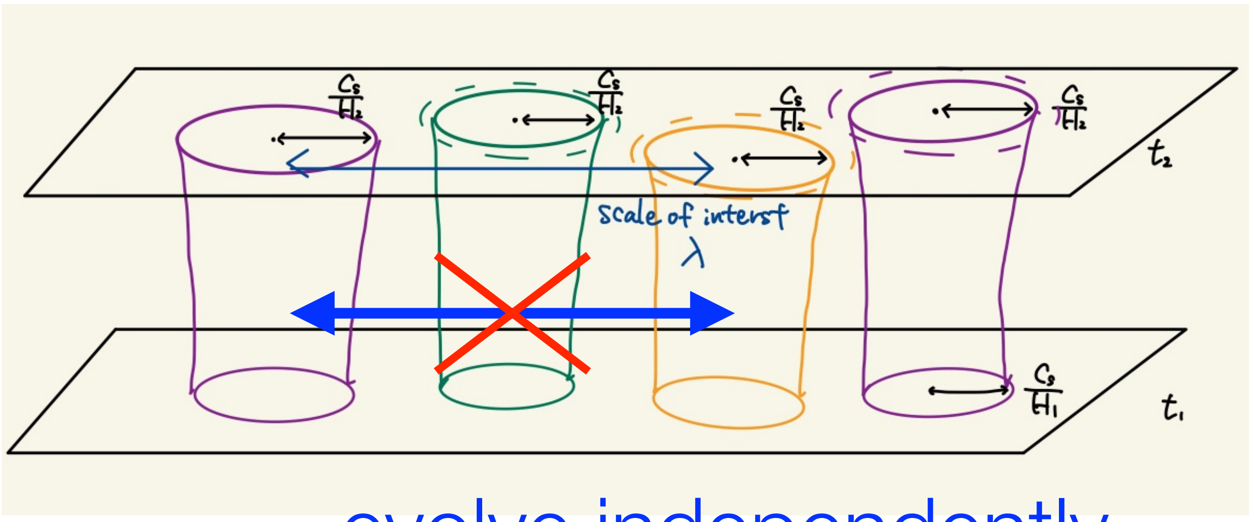
=

(☆)

t

late

early



evolve independently

When (☆) holds?

w/o gravity

w/o gravity (or for a fixed geometry)

【Locality】 ⟶ (☆)

e.g. canonical scalar field

$$\underbrace{\ddot{\phi} + 3H\dot{\phi}}_{\sim H^2} + dV/d\phi + \underbrace{\Delta}_{\sim k^2/a^2 \to O(\varepsilon^2)}\phi + \cdots = 0$$

Δ: Laplacian

We can ignore spatial gradient, when the corse grained field only includes k/a < H → smoothing scale λ_s > 1/H → small parameter ε =k/aH << 1

~~【Locality】~~ $\not\longrightarrow$ (☆)

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} + \mu(\Delta - \mu^2)^{-1}[\nabla_i\phi\nabla^i\phi] = 0$$

w/ gravity

$$ds^2 = -\alpha^2 dt^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \qquad g_{ij} = e^{2\psi}\gamma_{ij}$$

【Locality】 $\not\longrightarrow$ (☆)

$$S = \int d^{d+1}x\, \alpha\sqrt{g}\mathcal{L} = \int d^{d+1}x\, \alpha\sqrt{g}\,[\mathcal{L}_{\rm g} + \mathcal{L}_{\rm matter}]$$

HC $\quad \mathscr{H} \equiv \dfrac{\partial(\alpha\sqrt{g}\mathscr{L})}{\partial\alpha} = 0 \quad$ local gauge const.

MC $\quad \mathscr{H}_i \equiv \dfrac{\partial(\alpha\sqrt{g}\mathscr{L})}{\partial\beta^i} = 0 \quad$ non-local gauge const.

$$\partial_i(\cdots) + (\cdots)\partial_i(\cdots) = 0$$

⟶ Solving $\mathcal{H}_i$ =0 gives rise to non-local terms.

e.g.,single scalar field, β_i=0 + $\boldsymbol{\delta\phi}$=0

$\mathcal{H}_i = 0 \rightarrow \boldsymbol{\alpha} = \Delta^{-1}(\ldots\ldots) + (\ldots\ldots)$

w/ gravity

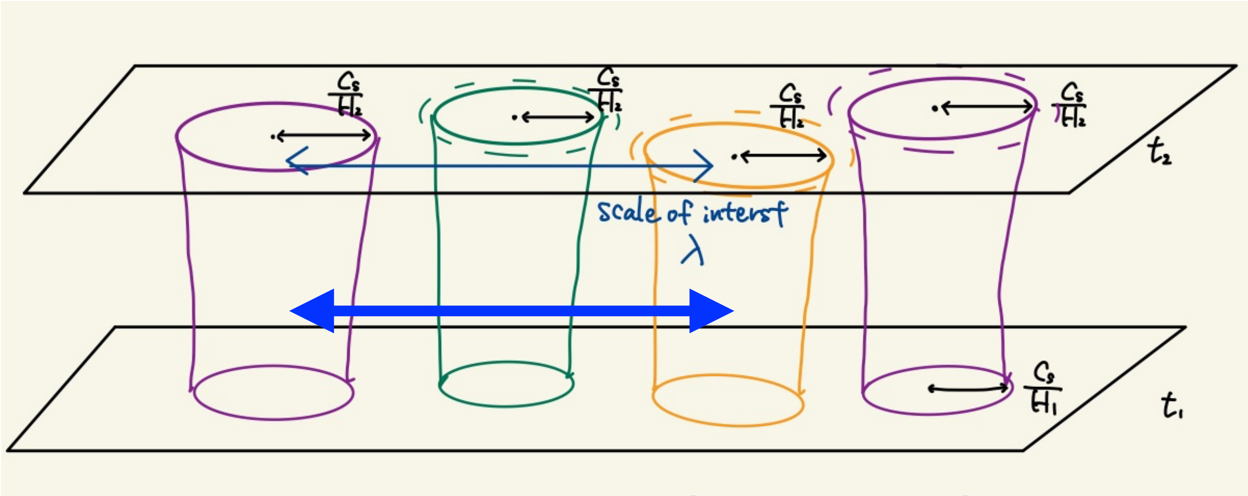
a local theory w/gravity

fields in $\mathcal{L}$ $\longrightarrow$ physical fields $\{\varphi_{phys}\}$

solving all constraints

non-local

e.g., MC $\mathscr{H}_i \equiv \partial(\alpha\sqrt{g}\mathscr{L})/\partial\beta^i = 0$ $\partial_i(\cdots)+(\cdots)\partial_i(\cdots)=0$



acausal interaction

$\xcancel{(\star)}$

Generalization

Tanaka & Y.U. (2021)

【Locality】

The Lagrangian is determined by a local function of coarse-grained field $\{\varphi^a\}$, i.e., $\mathcal{L}(x) = \mathcal{L}\left[\{\varphi^a(x)\}\right]$

↔ “Non-locally” appears only appear by solving gauge constraints.

【sDiff】

The action remains invariant under the spatial Diff.

$$x^i \to \tilde{x}^i(t, \mathbf{x})$$

➡ Solving MC does not cause any accusal issues.

Validity of (☆) at the leading order of the gradient exp.

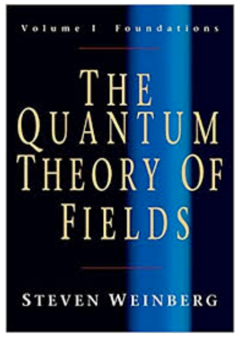
* The coarse-grained fields (metric & matter fields) should be x-independent.

Excluding Solid configuration, Spatially non-flat FLRW as 0th order of gradient exp.

For perturbative including of curvature see Artigas, Shi, Tanaka (2024)

Recall QED

from



Volume I Foundations

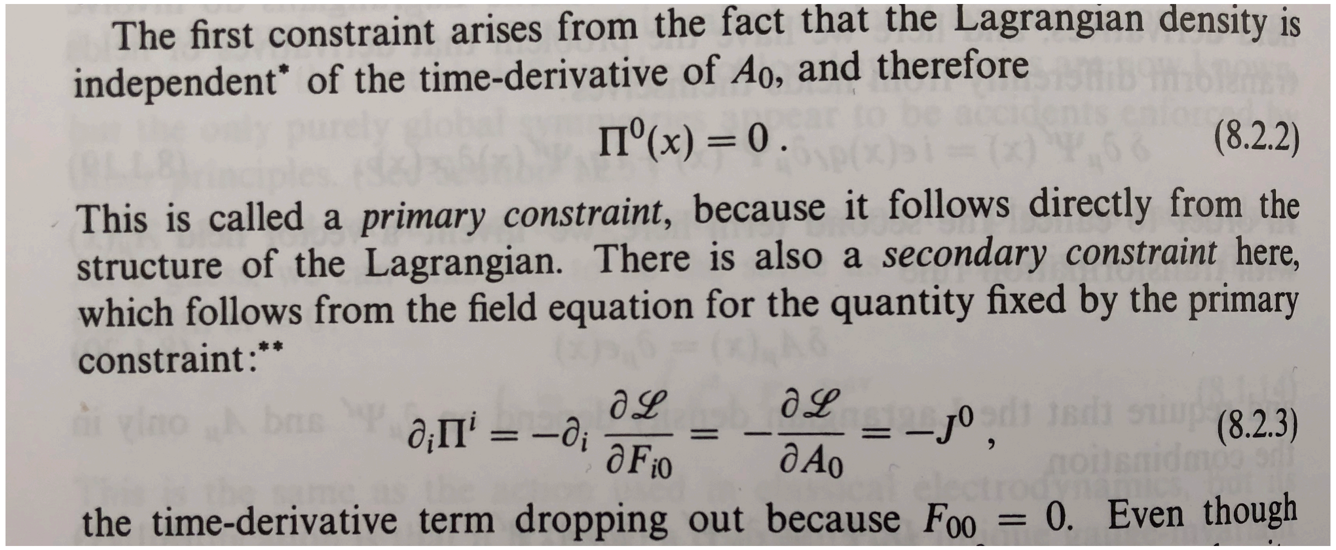
THE QUANTUM THEORY OF FIELDS

STEVEN WEINBERG

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

+ matter fields

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$



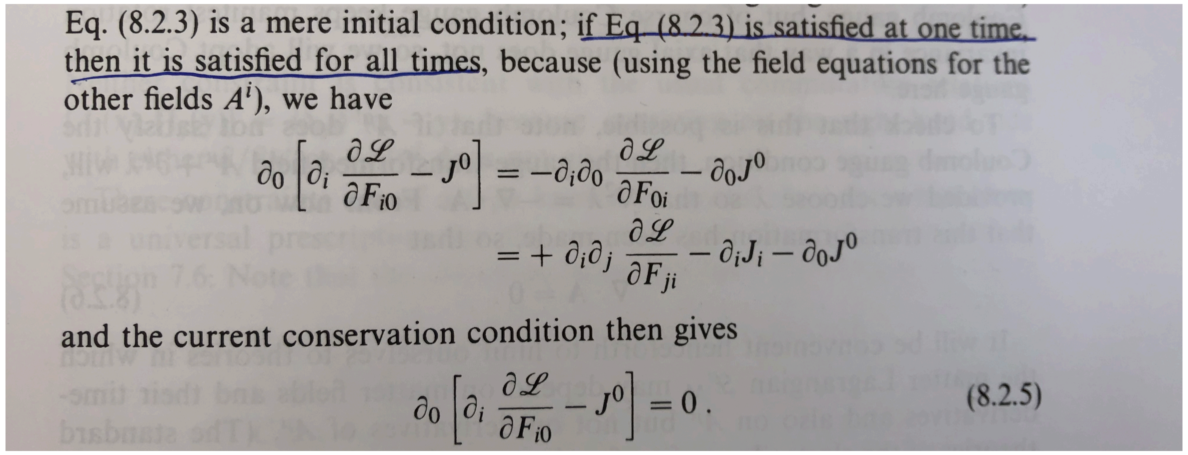
The first constraint arises from the fact that the Lagrangian density is independent* of the time-derivative of A_0, and therefore

$$\Pi^0(x) = 0\,. \qquad (8.2.2)$$

This is called a *primary constraint*, because it follows directly from the structure of the Lagrangian. There is also a *secondary constraint* here, which follows from the field equation for the quantity fixed by the primary constraint:**

$$\partial_i \Pi^i = -\partial_i \frac{\partial \mathscr{L}}{\partial F_{i0}} = -\frac{\partial \mathscr{L}}{\partial A_0} = -J^0\,, \qquad (8.2.3)$$

the time-derivative term dropping out because $F_{00} = 0$. Even though



Eq. (8.2.3) is a mere initial condition; if Eq. (8.2.3) is satisfied at one time, then it is satisfied for all times, because (using the field equations for the other fields A^i), we have

$$\begin{aligned}\partial_0 \left[\partial_i \frac{\partial \mathscr{L}}{\partial F_{i0}} - J^0\right] &= -\partial_i \partial_0 \frac{\partial \mathscr{L}}{\partial F_{0i}} - \partial_0 J^0 \\ &= +\,\partial_i \partial_j \frac{\partial \mathscr{L}}{\partial F_{ji}} - \partial_i J_i - \partial_0 J^0\end{aligned}$$

and the current conservation condition then gives

$$\partial_0 \left[\partial_i \frac{\partial \mathscr{L}}{\partial F_{i0}} - J^0\right] = 0\,. \qquad (8.2.5)$$

If a gauge constraint holds at one time, it holds at any time.

The same argument is also possible for other gauge const e.g., U(1) gauge const.

Derivation

Tanaka & Y.U. (2021)

Spatial Diff invariance

$$x^i \to e^{\xi^\mu \partial_\mu} x^i = x^i + \xi^i + \cdots \qquad \xi^\mu = (0,\, \xi^i(t,\, \boldsymbol{x}))$$

$$0 = \delta_\xi S = \int d^{d+1}x \left\{ \xi^i \partial_t \left(\sqrt{g}\mathcal{H}_i \right) + \sum_a{}' \frac{\delta S}{\delta \varphi^a} \underset{\uparrow}{\delta_\xi \varphi^a} + \mathcal{O}(\epsilon^2) \right\}$$

Other fields than lapse and shift

$$\delta_\xi \alpha = 0, \qquad \delta_\xi \beta^i = -\,\dot{\xi}^i + \cdots$$

* bdry term vanishes since $\mathcal{H}_i$ is a generator of sDiff.

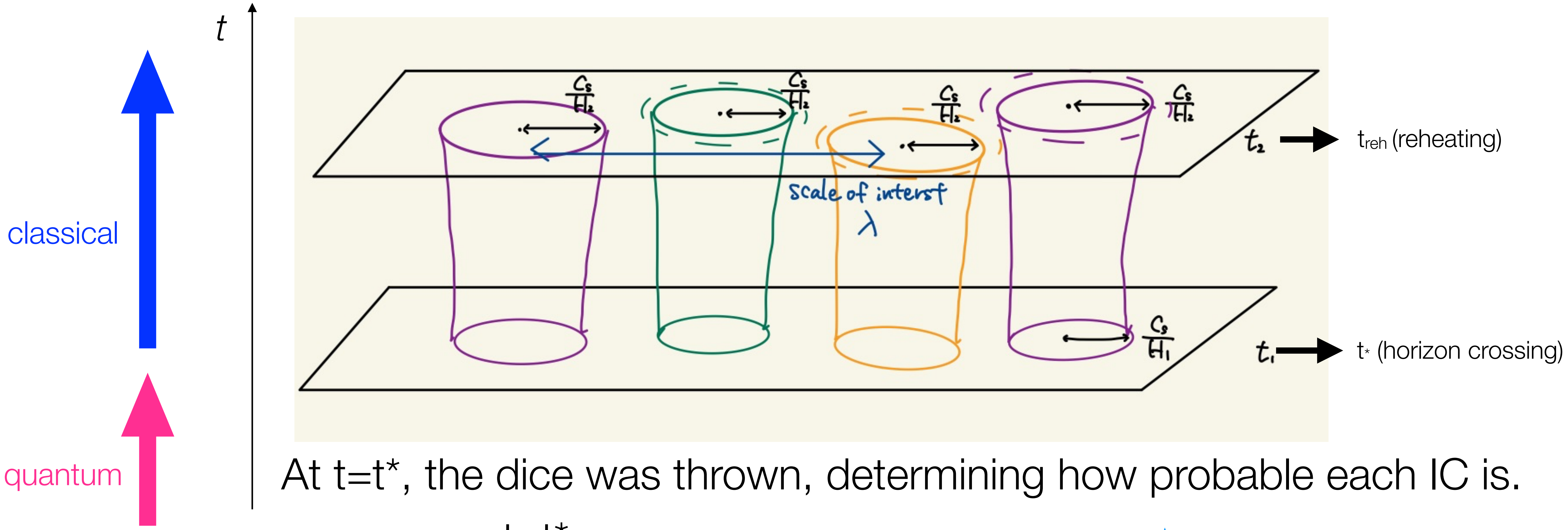
Momentum constraint $\mathscr{H}_i \equiv \alpha \partial \mathscr{L} / \partial \beta^i$

- $\mathcal{H}_i = 0$ at $t=t_i$ $\qquad$ $\mathcal{H}_i = 0$ at all t
- Even if not, $\qquad$ $\mathcal{H}_i = \mathcal{C}_i / \sqrt{g} \propto 1/V_{\rm phys}$

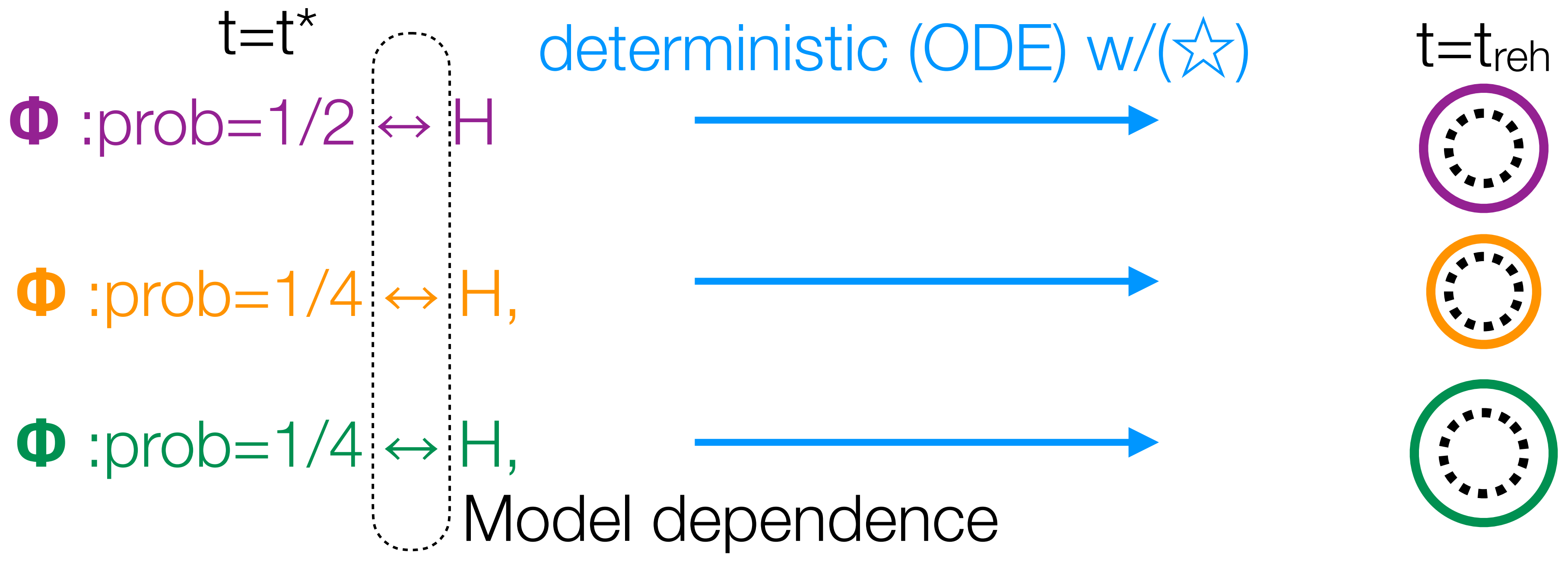
Generalization of Sugiyama, Komatsu, & Futamase (12) Garriga, Y.U., & Vernizzi (16)

δN formalism

Starobinsky (82, 85),
Sasaki & Stewart (95),
Sasaki & Tanaka (98),



At t=t*, the dice was thrown, determining how probable each IC is.



t=t*

deterministic (ODE) w/(☆)

$t=t_{reh}$

Φ :prob=1/2 ↔ H

Φ :prob=1/4 ↔ H,

Φ :prob=1/4 ↔ H,

Model dependence

H: expansion

IC which satisfies MC

PDF of ζ at t_{reh} from PDF of ϕ^I at t*

Our generalization is naively….

Tanaka & Y.U. (2021)

delta N is mapping between the initial input (→ inhogegeneity) and the corresponding e-folding number

For a single field model of inflation

Non-local form (from MC)

IC at horizon crossing: $\underbrace{\phi^*, \pi_\phi^*, h_+^*, h_\times^*}_{\equiv\ \Phi^a_{\rm phys}{}^*\ (a=1,\ 2,\ 3,4)}$ + 3 shear DOFs $\sigma_1,\sigma_2,\sigma_3$

$N_i=0$ + TT at $t=t^*$

$$\Delta N(t_f, \mathbf{x}) = N(t_f, \underset{\Phi^a_{\rm phys}{}^*}{\underset{\downarrow}{\mathbf{x}}}) - \langle N(t_f)\rangle = \mathcal{F}[t; \{\Phi^a_{\rm phys}{}^*\}]$$

The mapping functional $\mathcal{F}$ becomes non-local.

~~Separate Universe~~

e.g. in ultra-slow roll modes

Our generalization is naively….

Tanaka & Y.U. (2021)

delta N is mapping between the initial input (→ inhogegeneity) and the corresponding e-folding number

For a single field model of inflation

Non-local form (from MC)

IC at horizon crossing: $$\underbrace{\underbrace{\phi^*, \pi_\phi^*, h_+^*, h_\times^*}_{\equiv \Phi^a_{\rm phys}{}^* \ (a=1,\ 2,\ 3,4)} \ + \ 3 \text{ shear DOFs } \sigma_1,\sigma_2,\sigma_3}_{\equiv \Phi^{a*} \ (a=1,\ \dots,\ 7)}$$

$N_i=0$ + TT at $t=t^*$

$$\Delta N(t_f, \mathbf{x}) = N(t_f, \underset{\substack{\downarrow \\ \Phi^{a*}}}{\mathbf{x}}) - \langle N(t_f) \rangle = \mathcal{F}[t; \{\Phi^{a*}\}]$$

Mapping function $\mathcal{F}$ becomes local, since equations to be solved are all local.

✔ Separate Universe is valid.

… while initial inputs $\sigma_i = \sigma_i(\Phi^a_{\rm phys}{}^*)$ becomes non-local.

Story in scalar field systems

Why we don't face non-locality in "usual" examples ??

GR + scalar fields

w/slow-roll Sugiyama, Komatsu, & Futamase(12)

w/o slow-roll Garriga, Y.U., & Vernizzi(16)

MC is ensured by HC w/ exponentially decaying error.

$$\mathcal{H}_i - \partial_i \mathcal{H} = (\cdots)/V_{\rm phys}$$

K : Expansion

${A^i}_j$: Shear

e.g. single canonical scalar field in GR

HC $$-\frac{3}{2}K^2 + {A^i}_j{A^j}_i + 16\pi G\rho = \mathcal{O}(\epsilon)$$

MC $$-\frac{3}{2}\partial_i K + \nabla_j {A^j}_i - 16\pi G\frac{\dot{\phi}}{\alpha}\partial_i\phi = \mathcal{O}(\epsilon^2)$$

(i, j) traceless shear $${A^i}_j = (\cdots)/V_{\rm phys}$$

The same argument does not work in the presence of vector/tensor fields.

Shear as $\mathcal{O}(\epsilon^0)$

Shear, $A^i{}_j$, has been assumed to be sub-leading order of gradient expansion.

$$ds^2 = -\alpha^2 dt^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

e.g, Lyth, Malik, & Sasaki (04)

$$\beta_i = O(\epsilon)\,, \quad \dot{\gamma}_{ij} = O(\epsilon^2) \longrightarrow A^i{}_j = \mathcal{O}(\epsilon)$$

Sometimes this can cause a problem (e.g., in USR)…

canonical single field (δϕ=0)

$$\zeta_k \simeq C_1(k) + C_2(k) \int \frac{dt}{a^d\,\varepsilon}$$

(*k/aH* <<1)

Weinberg's adiabatic mode Weinberg's second mode

Sasaki & Tanaka (99)

$$\int dt\,(\cdots) \times (\text{shear})$$

Garriga, Y.U., & Vernizzi (16)

Tanaka & Y.U. (21)

To have 2 independent solutions at $\mathcal{O}(\epsilon^0)$, shear has to be $\mathcal{O}(\epsilon^0)$

Dropping the shear can also cause an inconsistency in MC.

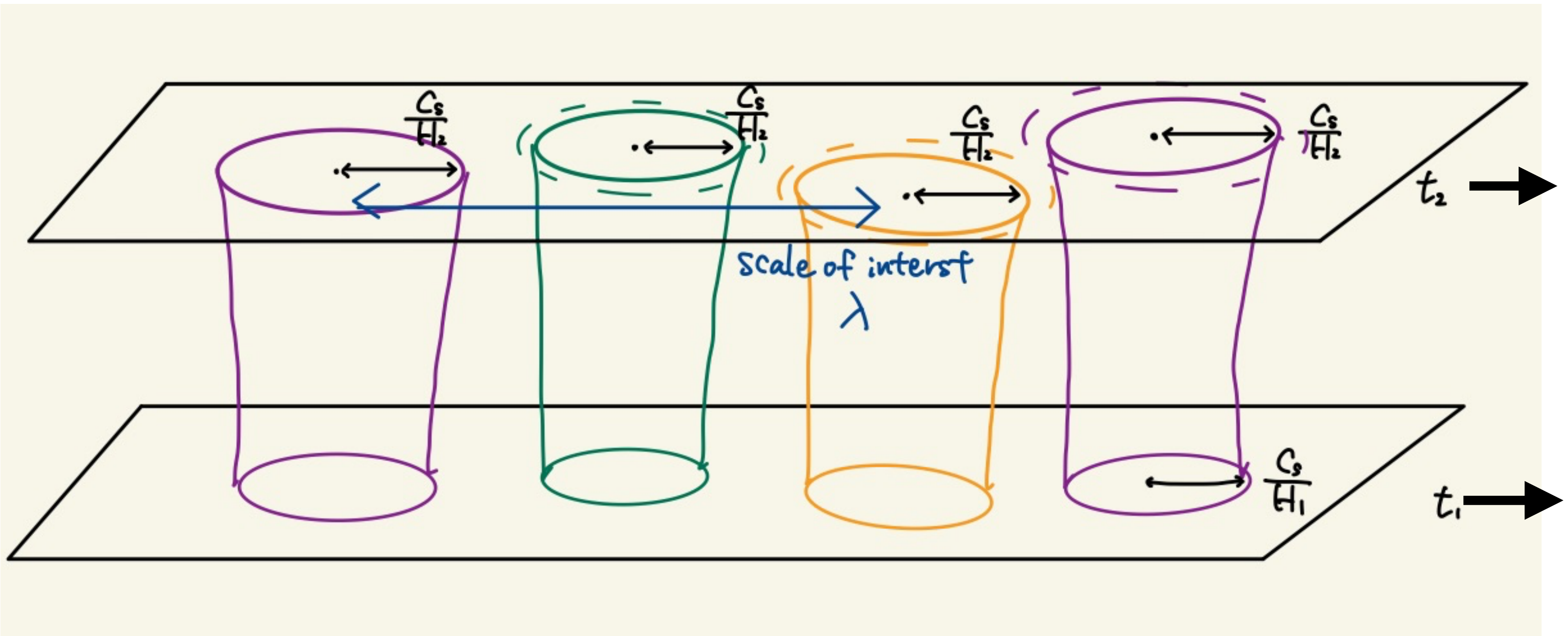
Generalized δN formalism for PGW

Tanaka & Y.U. (2021,2023, 2024)

t

classical

quantum



→ t_{reh} (reheating)

→ t_* (horizon crossing)

At t=t*, the dice was thrown, determining how probable each IC is.

t=t*

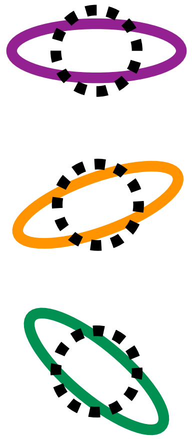
deterministic (ODE)

$t=t_{reh}$

Φ :prob=1/2 ↔ σ

Φ :prob=1/4 ↔ σ

Φ :prob=1/4 ↔ σ



σ: shear

Model dependence

IC which satisfies MC

PDF of GW at t_{reh} from PDF of **Φ** at t*

An application

Generalized δN formalism

D scalar fields ϕ^I (incl. inflaton) + D' U(1) gauge fields $A_{i(\alpha)}$

$X^{IJ} \equiv -\partial_\mu \phi^I \partial^\mu \phi^J / 2$

$$\mathcal{L}_{\rm mat} = \mathcal{P}(X^{IJ}, \phi^I) - \sum_{\alpha=1}^{D'} \frac{f^2_{(\alpha)}(X^{IJ}, \phi^I)}{4} F_{(\alpha)\,\mu\nu} F^{\mu\nu}_{(\alpha)}$$

+CS term

(Sub-leading at large scales)

+ general relativity

$$\gamma_{ij}(t, \boldsymbol{x}) = \gamma_{ij*}(\boldsymbol{x}) - 2\left[\gamma_{il*}(\boldsymbol{x})\gamma_{jm*}(\boldsymbol{x})\right]^{\rm TL} \sum_{\alpha=1}^{D'} \underline{\pi^l_{(\alpha)}(\boldsymbol{x})\pi^m_{(\alpha)}(\boldsymbol{x})\,\mathcal{I}_{(\alpha)}(t; \{\varphi^{a'}_*\}')} + \mathcal{O}(\epsilon)$$

$$\sim \frac{\pi^l_{(\alpha)}}{|\pi_{(\alpha)}|} \frac{\pi^m_{(\alpha)}}{|\pi_{(\alpha)}|} \int dN \, \frac{\rho_{A(\alpha)}}{\rho_{\rm tot}}$$

Spatial coord. $g_{0i}=0$

t x (Lapse) → t

$$\mathcal{I}_{(\alpha)}(t; \{\varphi^{a'}_*\}') = \frac{1}{M^2_{\rm pl}} \int_{t_*}^{t} \frac{dt'}{e^{3\psi(t')}} \int_{t_*}^{t'} \frac{dt''}{e^{\psi(t'')} f^2_{(\alpha)}(X^{IJ}(t''), \phi^I(t''))}$$

$\pi^i_{(\alpha)}$: Conjugate momentum of $A_{i(\alpha)}$

Maxwell equation $\partial_t \pi^i_{(\alpha)} = \mathcal{O}(\epsilon)$

Applicable to fully non-linear perturbations!

Model independent properties: Order estimation

$$\gamma_{ij}(t,\boldsymbol{x}) = \gamma_{ij*}(\boldsymbol{x}) - 2\left[\gamma_{il*}(\boldsymbol{x})\gamma_{jm*}(\boldsymbol{x})\right]^{\rm TL}\sum_{\alpha=1}^{D'}\underbrace{\pi^l_{(\alpha)}(\boldsymbol{x})\pi^m_{(\alpha)}(\boldsymbol{x})\,\mathcal{I}_{(\alpha)}(t;\{\varphi^{a'}_*\}')}_{\sim\ \frac{\pi^l_{(\alpha)}}{|\pi_{(\alpha)}|}\frac{\pi^m_{(\alpha)}}{|\pi_{(\alpha)}|}\int dN\,\frac{\rho_{A(\alpha)}}{\rho_{\rm tot}}} + \mathcal{O}(\epsilon)$$

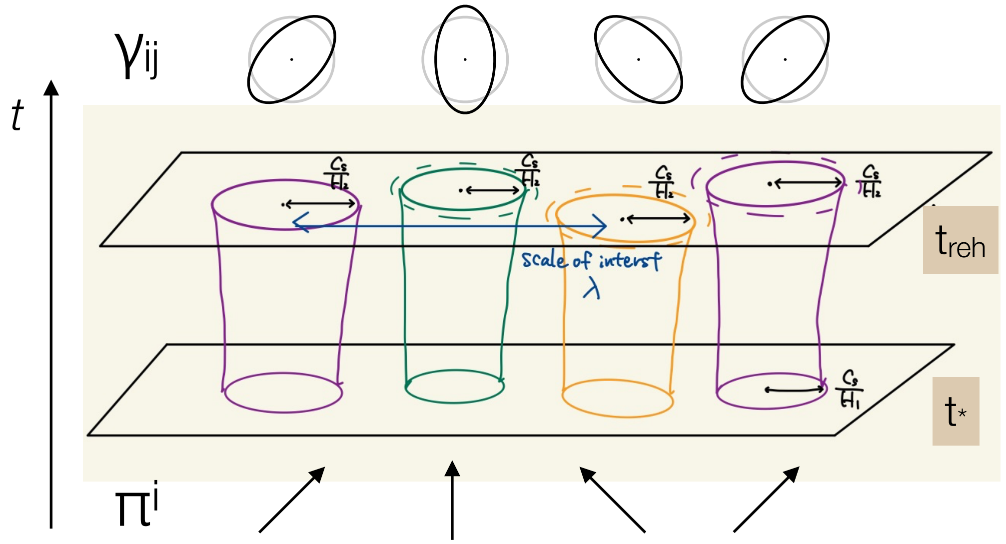
- When $\rho_{A(alpha)}/\rho_{tot}$ takes a large value during ΔN,

GW sourced by gauge fields becomes larger for a larger $\frac{\rho_A}{\rho_{\rm tot}}\Delta N$

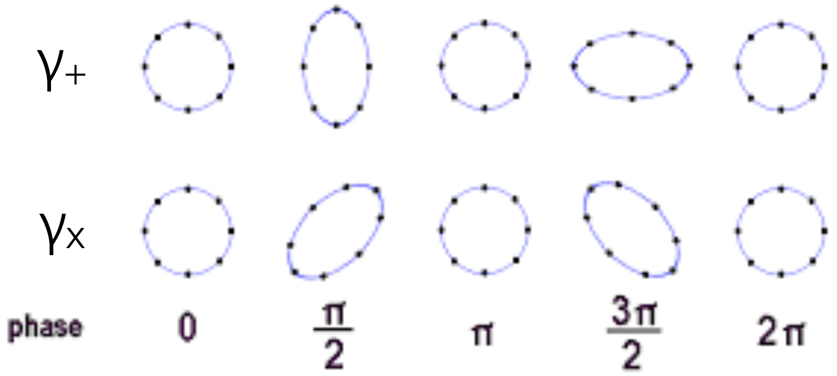
Namely for ΔN > O(1)

$$\delta\gamma(t_f) \sim \delta\gamma_* + \delta\left(\frac{\pi_{(\alpha)}}{|\pi_{(\alpha)}|}\right)\int dN\,\frac{\rho_{A(\alpha)}}{\rho_{\rm tot}}$$

$$\mathrm{X} \quad \sim \underbrace{\sqrt{\frac{\rho_{\rm tot}}{\rho_{A(\alpha)*}}}}_{\text{initial seed amp.}}\ \underbrace{\frac{\rho_{A(\alpha)}}{\rho_{\rm tot}}\Delta N}_{\text{conversion to }\delta\gamma}$$



Model independent properties : Linear polarization



Condition for γ_+ and $\gamma_\times$ to have different spectrums?

$$\gamma_{ij}(t,\boldsymbol{x}) = \gamma_{ij*}(\boldsymbol{x}) - 2\left[\gamma_{il*}(\boldsymbol{x})\gamma_{jm*}(\boldsymbol{x})\right]^{\mathrm{TL}} \sum_{\alpha=1}^{D'} \pi^{l}_{(\alpha)}(\boldsymbol{x})\pi^{m}_{(\alpha)}(\boldsymbol{x})\mathcal{I}_{(\alpha)}\left[\phi^I\right] \quad + \mathcal{O}(\epsilon)$$

$$\uparrow$$

$$\delta\mathcal{I}_{(\alpha)} \sim \frac{\partial\mathcal{I}_{(\alpha)}}{\partial\phi^I}\delta\phi^I_{\text{intrinsic}} + \boxed{\frac{\partial\mathcal{I}_{(\alpha)}}{\partial\rho_{A(\alpha)}}\delta\rho_{A(\alpha)}}$$

Origin of linear polarization

- A large scale evolution ($t_* \leqq t \leqq t_{\mathrm{reh}}$) can generate the linear polarization, when the backreaction of gauge fields on scalar fields ϕ^I become important.

$$\phi^I = \phi^I_{\text{intrinsic}} + \phi^I_{\text{sourced}}[\rho_{A(\alpha)}]$$

See also Fujita et al. (18)

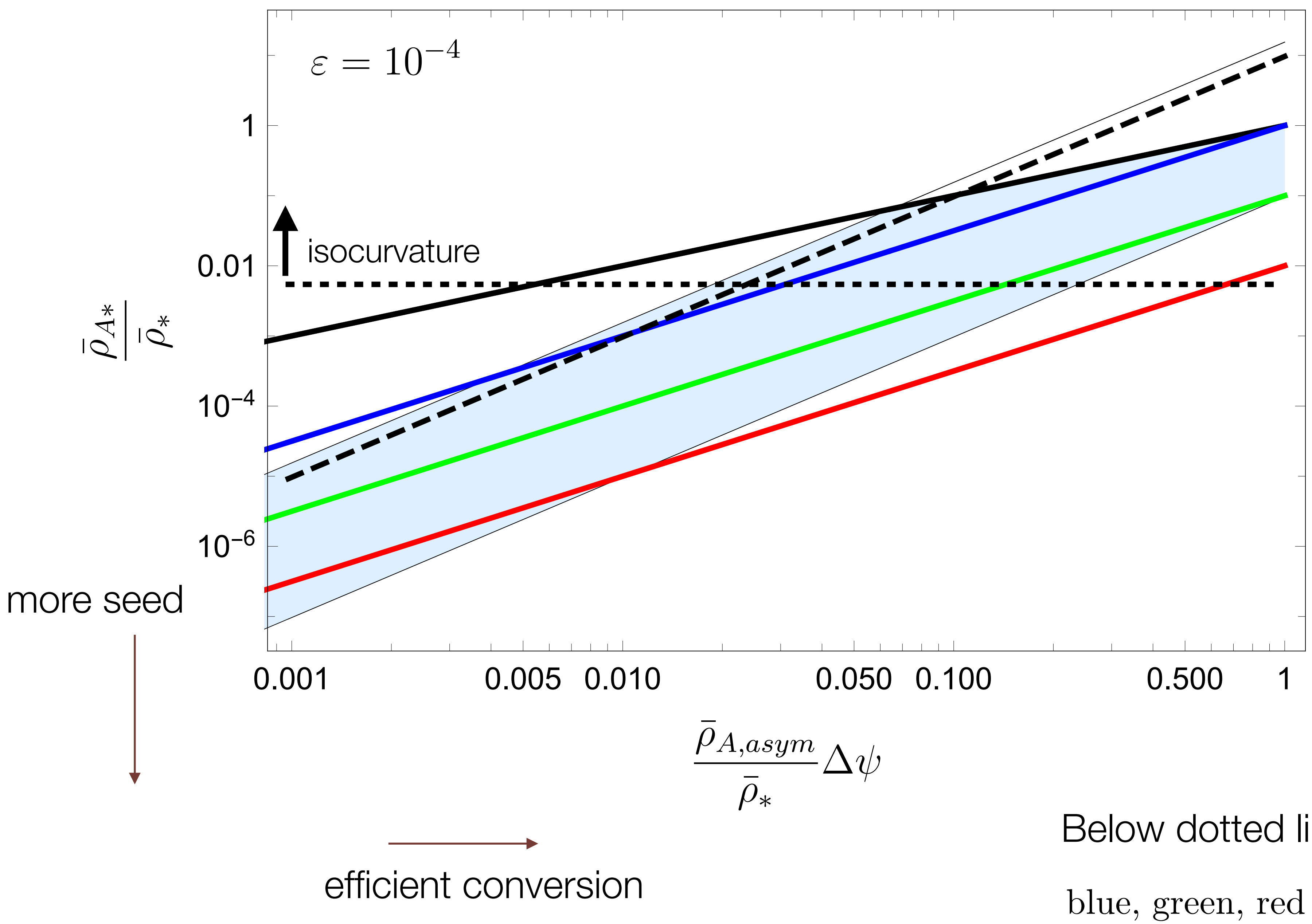
increasing gauge fields start to modify the evolution of ϕ^I

Tanaka & Y.U. (2023, 2024)

Prospects on future CMB experiments

Test of Cosmological Principle from PGW observations

inflaton+spectator+1 gauge field



more seed

efficient conversion

GW detectable $P_\gamma/P_\zeta > 10^{-3}$

Simons Observatory (2024~)
$\sigma(r) \sim 10^{-3}$

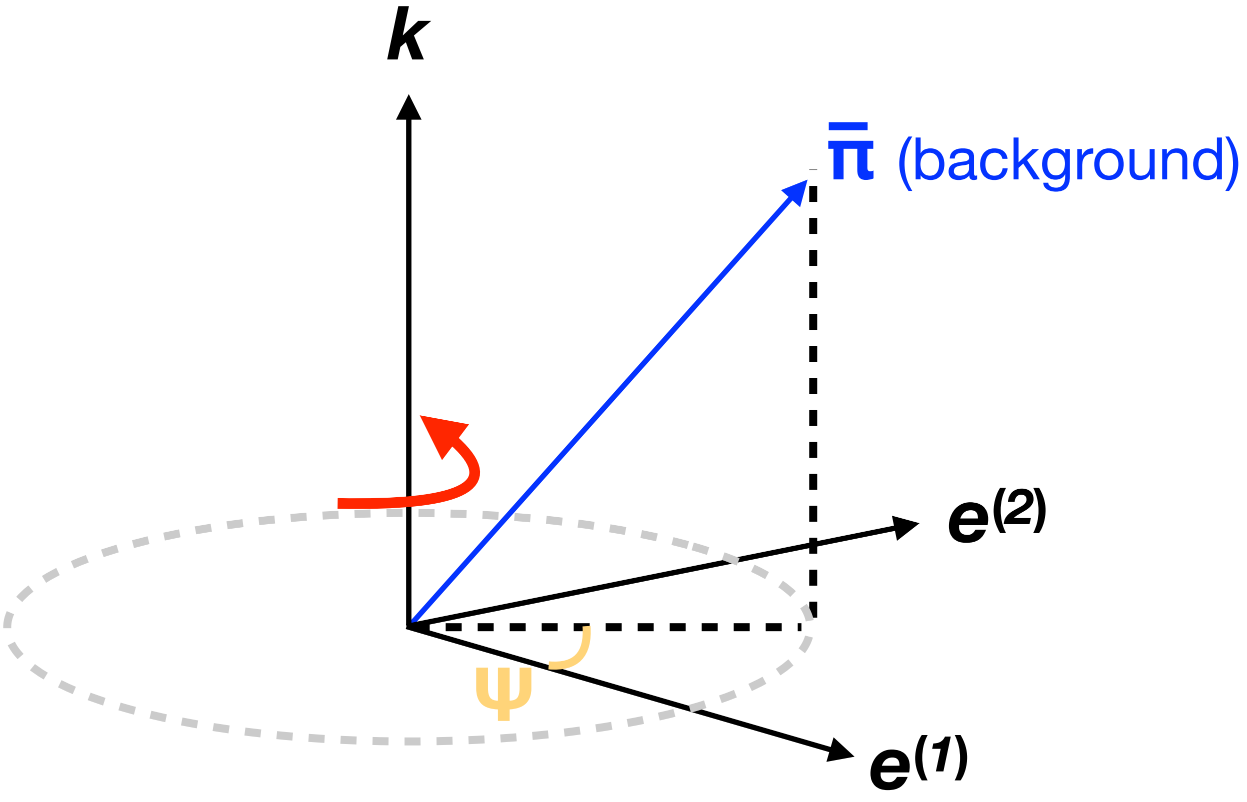
Bound from <ζζ>,<ζζζ>,…

Below dotted line → O(1) statistical anisotropy in GW

blue, green, red lines correspond to $f_{\rm NL}^{\gamma,A} = 1,\ 10^2,\ 10^4$

Imprints of multi-gauge fields

Tanaka & Y.U. (2023, 2024)



$<\zeta\,\gamma^{\times}>, <\gamma^{+}\,\gamma^{\times}> \propto \sin\Psi$

For single gauge field models, one can choose Ψ=0

$<\zeta\,\gamma^{\times}>=<\gamma^{+}\,\gamma^{\times}>=0$

Soda, Kanno, & Watanabe (10)

For multi gauge field models, one can choose $\Psi_1 = 0$, yet In general $\Psi_2, \Psi_3, \ldots \neq 0$

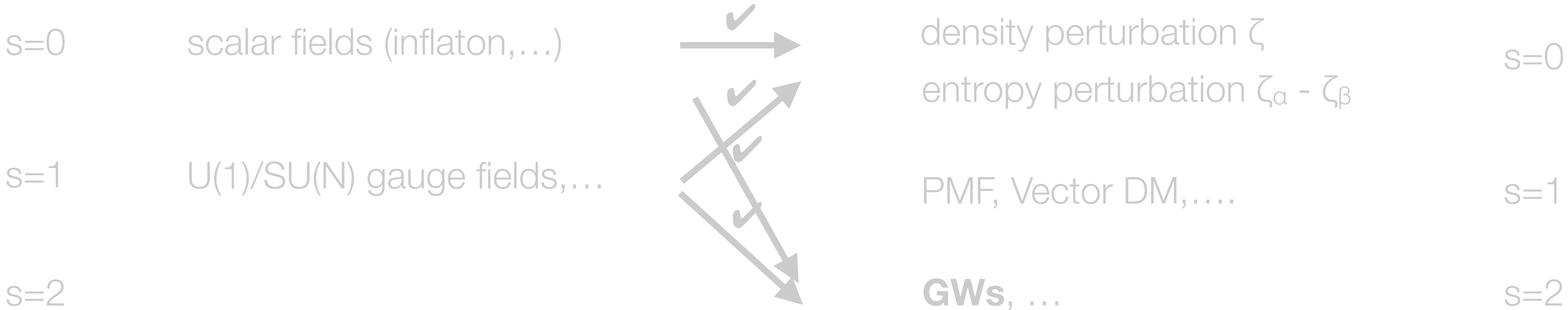
$<\zeta\,\gamma^{\times}>, <\gamma^{+}\,\gamma^{\times}> \neq 0 \;\rightarrow\; <\zeta\,\gamma^{+}> \neq <\zeta\,\gamma^{-}>$

Counterpart of isocurvature for light scale fields

Outline

1) Generalization of separate universe and delta N

- From arbitrary (integer) spins to arbitrary spins Tanaka & Y.U. (2021, 2023, 2024)



s=0 scalar fields (inflaton,…) density perturbation ζ s=0

entropy perturbation ζ_α - ζ_β

s=1 U(1)/SU(N) gauge fields,… PMF, Vector DM,…. s=1

s=2 **GWs**, … s=2

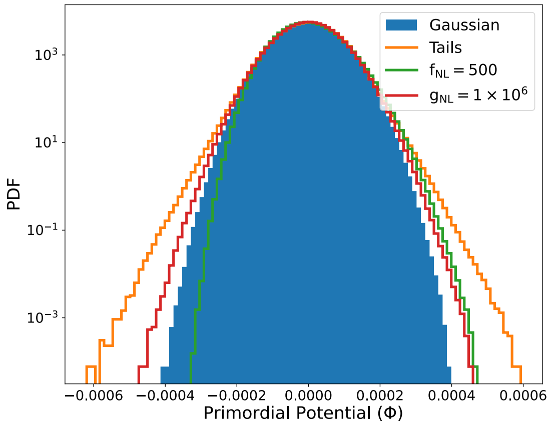
- Clarification of their validity From gradient expansion

2) Going beyond gradient expansion w/ Saha, Tada (in prep)

Scalar fields system where impacts of subhorizon modes can be potentially important. (USR model)

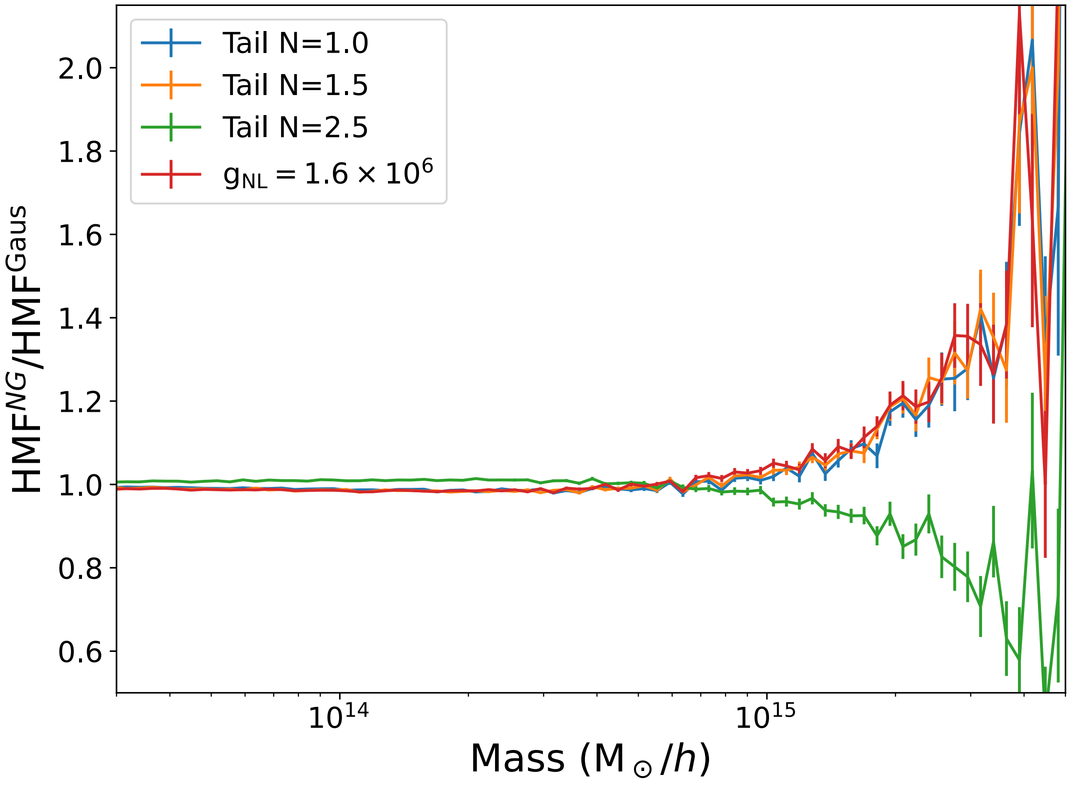
Incl. initial NG,…

Impact of non-Gaussian tail on halo mass function

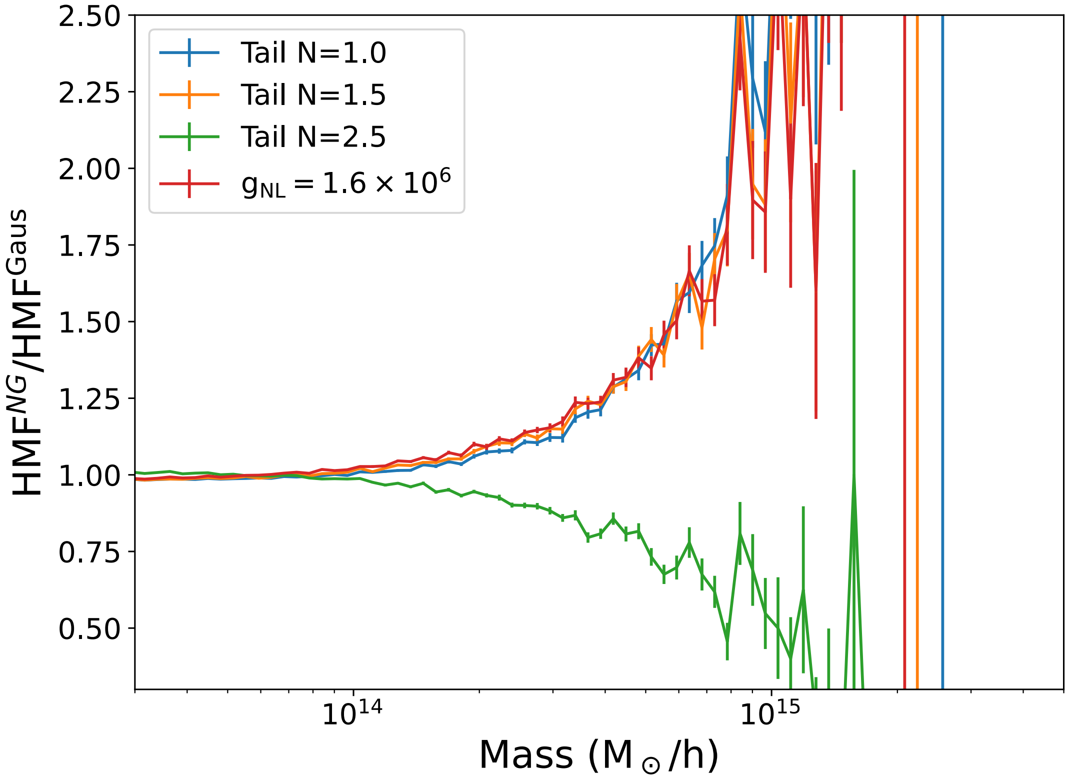


… and scale dependent bias, cold/hot spots in CMB, and so on

Coulton, Philcox, Villaescus-Navarro (24)



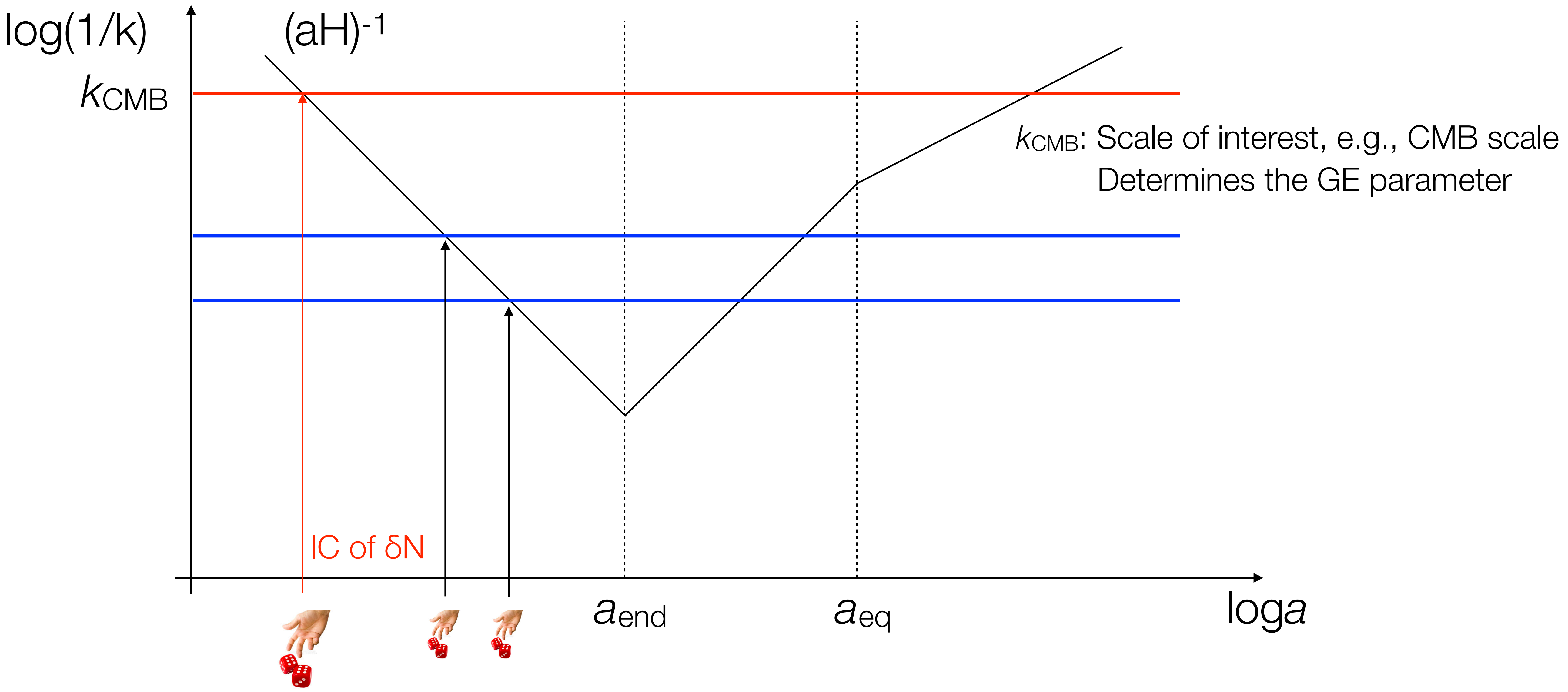
(a) $z = 0.0$



(b) $z = 1.0$

… yet $g_{\rm NL} = -5.8 \pm 6.5 \times 10^4$ (PLANCK18), so excluded 20σ

Diffusion effect



k_{CMB}: Scale of interest, e.g., CMB scale
Determines the GE parameter

Via the influence of later comers, the (superH) evolution becomes stochastic.
Not captured at the leading order of GE. For early stage work, see Y. Tanaka & Sasaki. (2006)

- Non-linear interactions (e.g., around the transition)
- Stochastic inflation from increasing superH modes

Stochastic approach

$$\phi = \phi_{\rm IR} + \phi_{\rm UV} \qquad \pi = \pi_{\rm IR} + \pi_{\rm UV}$$

k< σaH k> σaH

Langevin eq.

$$\dot{\phi}_{\rm IR} = \frac{1}{a^3}\pi_{\rm IR} + \xi_\phi$$

$$\dot{\pi}_{\rm IR} = -a^3 V_{,\phi}(\phi_{\rm IR}) + \xi_\pi$$

Fokker - Planck eq.

$$\frac{\partial}{\partial t}p(x,t) = -\frac{\partial}{\partial x}\left[\mu(x,t)p(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[D(x,t)p(x,t)\right].$$

Drift Diffusion

x → (ϕ, π)

ξ_ϕ, ξ_π follows the Gaussian statistics determined by subH modes (k> σaH)

Application of Lattice to USR

- **Lattice to provide IC of δN**

Caravano et al. (21, 22, 24)

Caravano, Fraciolini, Renaux-Petel (24, 25)

- Dynamics of matter fields (ϕ, A_μ) are fully non-perturbative.

- Lattice does not directly solving δN.

- No metric perturbation

- **Lattice to solve Fokker-Planck eq**

Mizuguchi, Murata, Tada (24)

- Lattice directly solves δN.
  - → Good for intuitive understanding likewise δN formalism
  - → Easy to capture the effects of beyond GE (eg diffusion) in δN

- Diffusion is assumed to follow Gaussian statistics at horizon crossing.

- No metric perturbation

Lattice-based δN beyond gradient expansion

(LattEfold)

with

Pankaj Saha (KEK, PD)

Yuichiro Tada (Rikkyo U.)

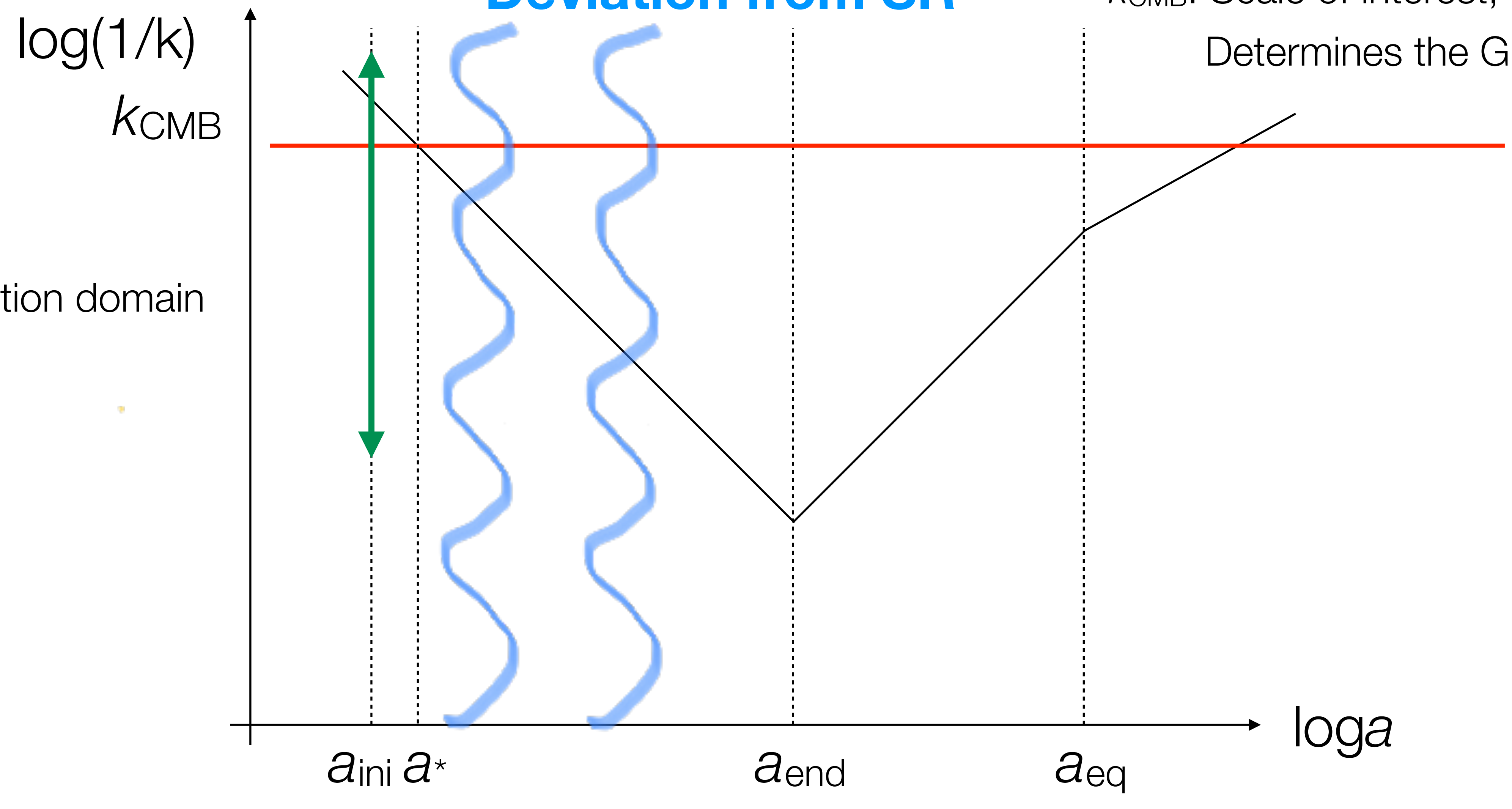
Diffusion effect

Saha, Tada, Y.U. (in prep)

Deviation from SR

$k_{\rm CMB}$: Scale of interest, e.g., CMB scale

Determines the GE parameter



Computation domain

$$d\bar{N} = \bar{H}dt$$

Bar: background

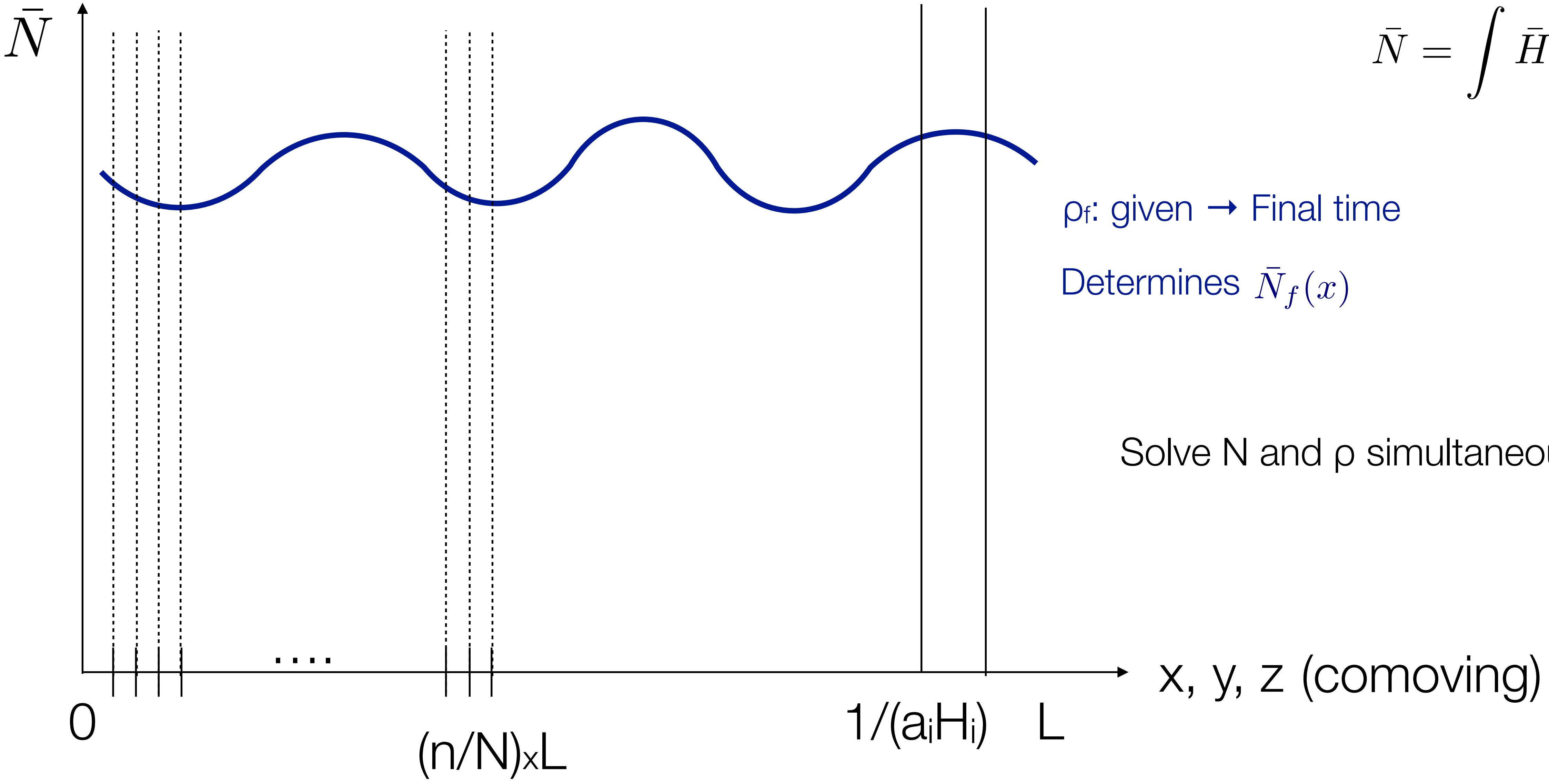
* Initial time is set during SR where initial Gaussian approx. holds well.

* Lattice is a classical simulation, so only tree level effects are included.

Uniform density slicing in lattice

Saha, Tada, Y.U. (in prep)

$$\bar{N} = \int \bar{H}\,dt$$



ρ_f: given → Final time

Determines $\bar{N}_f(x)$

Solve N and ρ simultaneously.

N=128

- $\zeta_f = \delta\bar{N}_f = \bar{N}_f[x] - \langle \bar{N}_f[x] \rangle$
- $\dfrac{d}{d\bar{N}} N(\bar{N}, x) = \dfrac{H(x, \bar{N})}{\bar{H}} \implies \zeta_f = N(\bar{N}_f(x), x) - \langle N(\bar{N}_f(x), x) \rangle$

Metric perturbation is only partly included. (Shear ignored.)

EOMs

Saha, Tada, Y.U. (in prep)

- (3+1)d Simulation of KG equation in perturbed flat FLRW w/$N(t,x) = \ln a(t,x)$

$$\frac{d\phi}{d\bar{N}} = \pi_\phi / \bar{H}$$

$$\frac{d\pi_\phi}{d\bar{N}} = -3\frac{H}{\bar{H}}\pi_\phi + \frac{\nabla^2\phi}{\bar{H}e^{2N}} - \frac{V_\phi}{\bar{H}}$$

$$H^2(t,x) = \frac{1}{3\mathrm{M}_p^2}\left(\frac{\pi_\phi^2}{2} + \frac{(\nabla\phi)^2}{2e^{2N}} + V(\phi)\right) \longrightarrow \qquad \bar{H}(t) = \langle H(t,x)\rangle$$

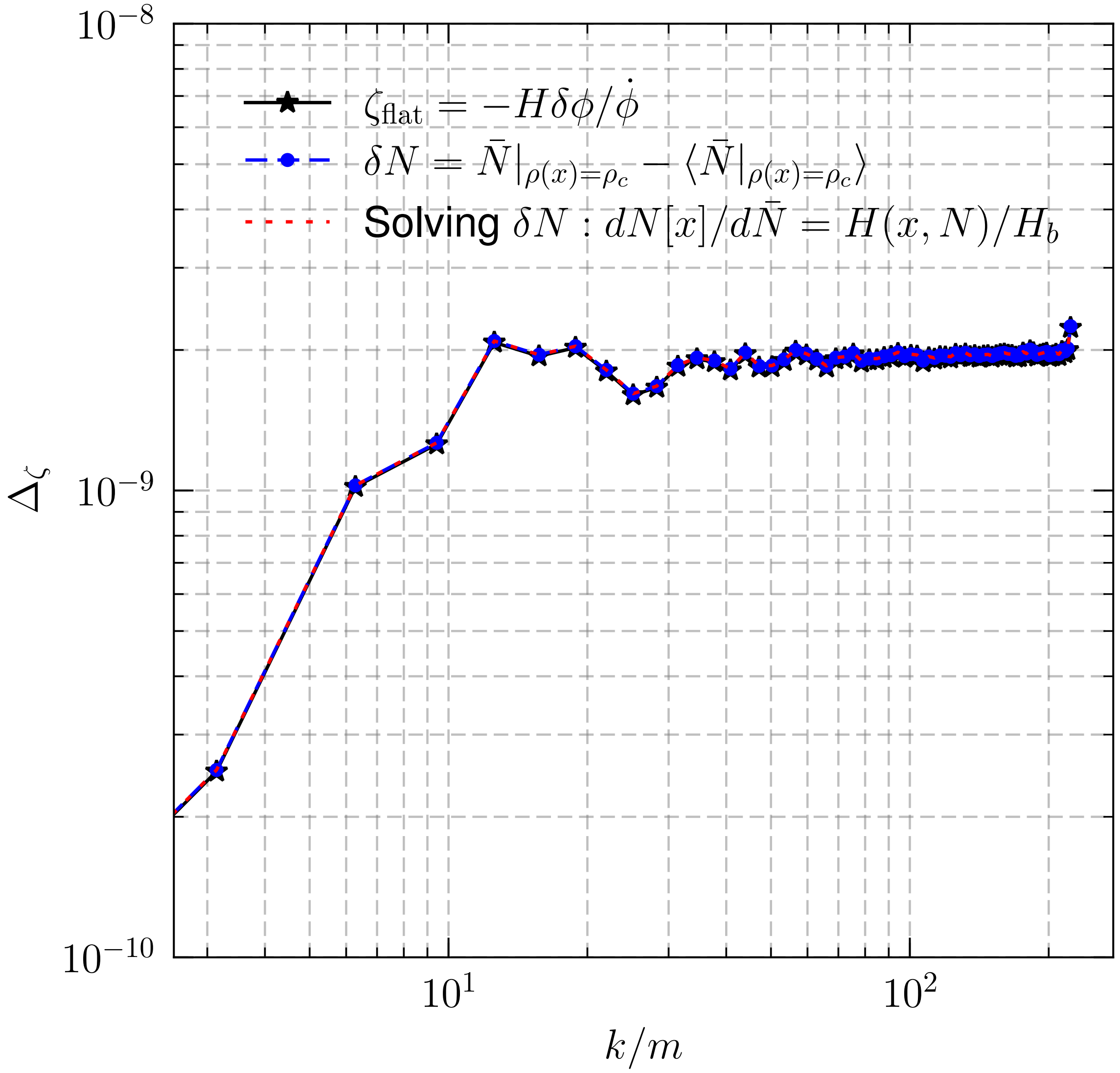
Note: Metric perturbation is not consistently taken into account.

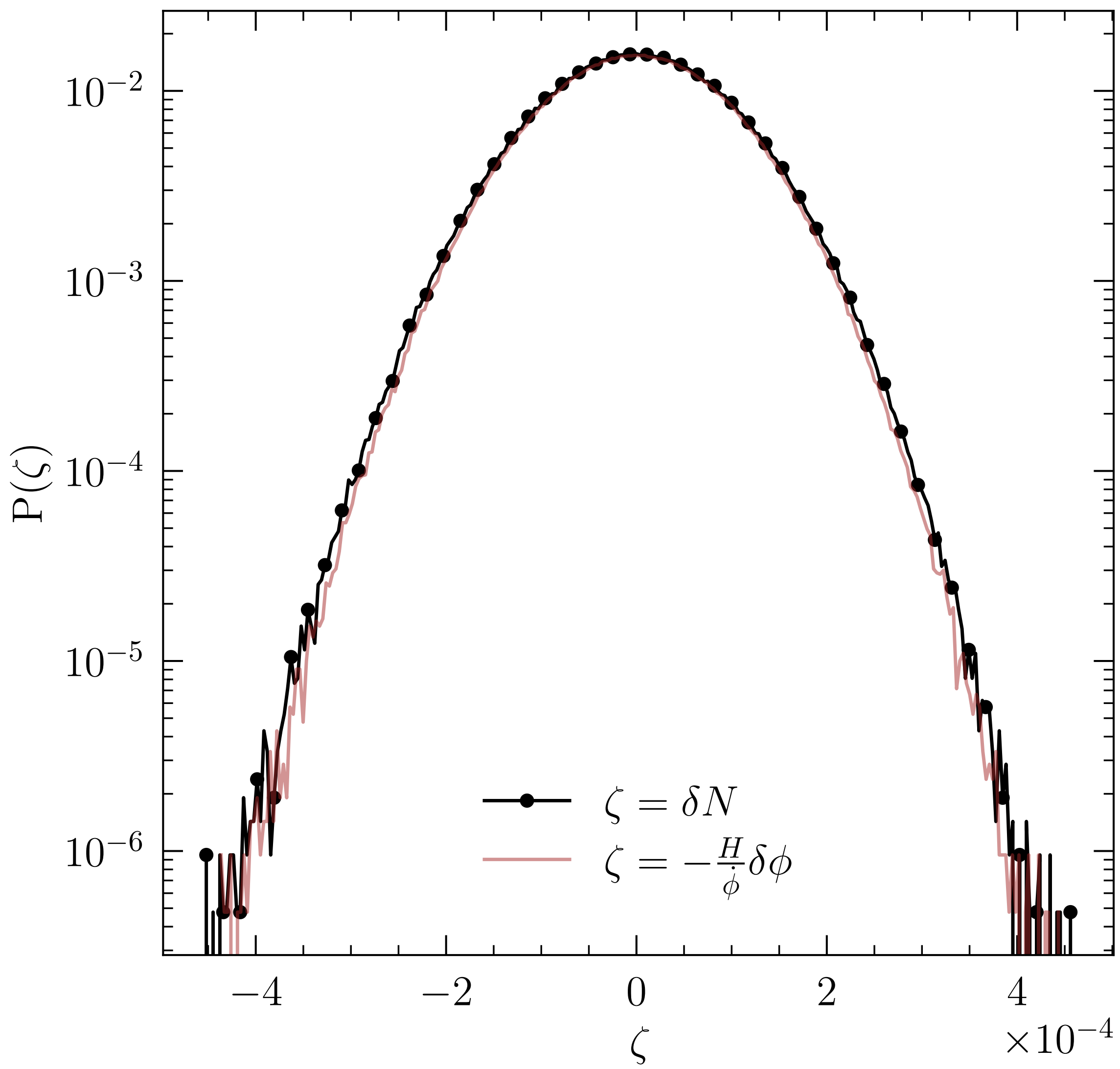
$g_{0i} = 0$ $\Bigg\{$

- Synchronous gauge, ignoring shear (or g_{ij} TL).
  - $\frac{d}{d\bar{N}}N(\bar{N},x) = \frac{H(x,\bar{N})}{\bar{H}} \implies \zeta_f = N(\bar{N}_f(x),x) - \langle N(\bar{N}_f(x),x)\rangle$
- δN gauge with $N(t,x) = \bar{N}(t)$ ignoring lapse and shear.
  - $\zeta_f = \delta\bar{N}_f = \bar{N}_f[x] - \langle\bar{N}_f[x]\rangle$

Results: $m^2\phi^2$ potential

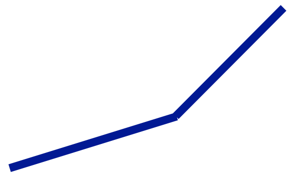
Saha, Tada, Y.U. (in prep)





$H_{ini} \sim 6m, L \sim 3/(a_{ini}H_{ini})$

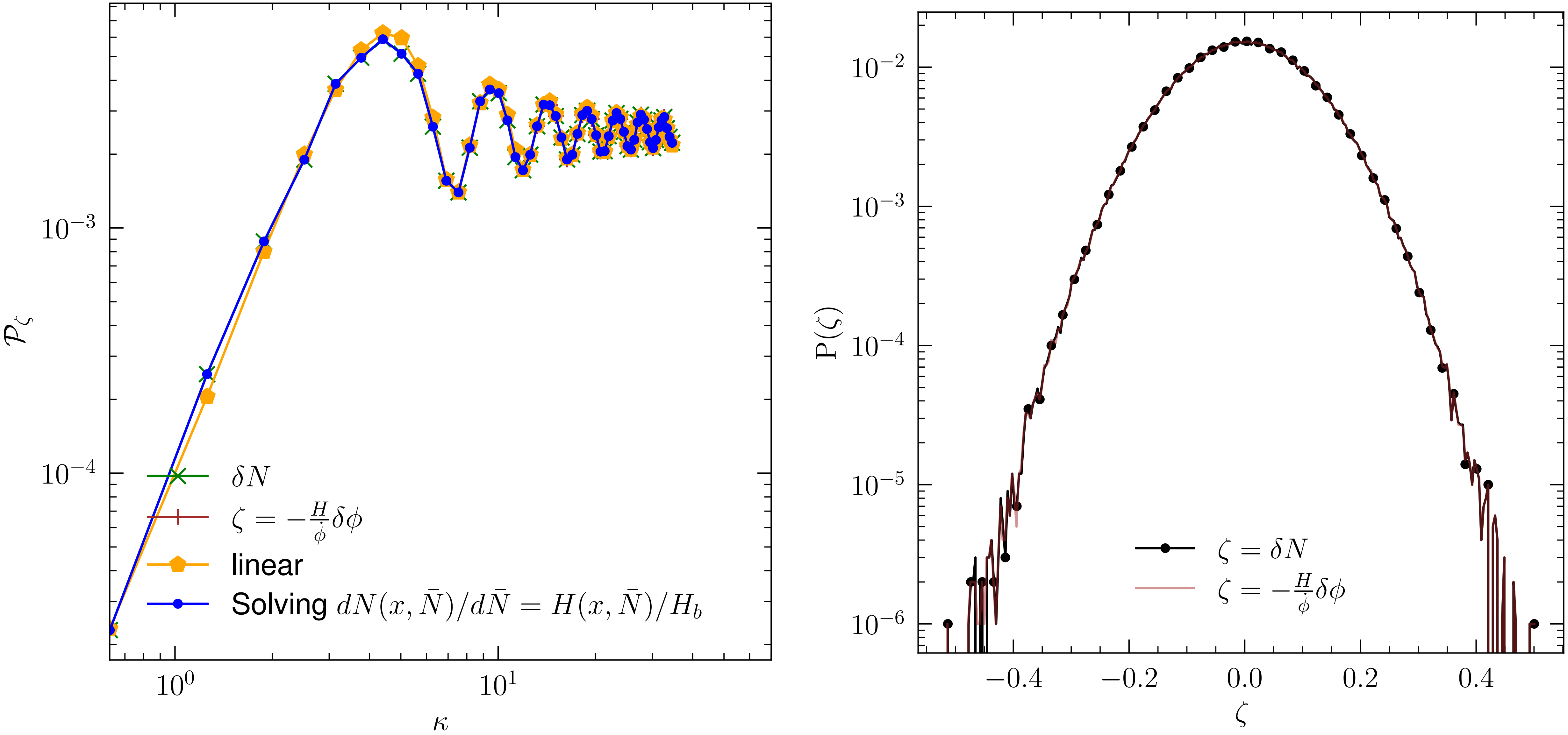
Results: Strobinsky linear potential



$$V(\phi) = \begin{cases} V_0 + A_+(\phi - \phi_0) & \text{for } \phi > \phi_0 \\ V_0 + A_-(\phi - \phi_0) & \text{for } \phi \leq \phi_0 \end{cases} \qquad (A_+ > A_-)$$

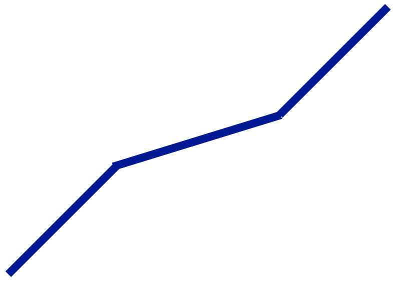
Starobinsky (92)

For analytic formula, see Pi & Wang (23)

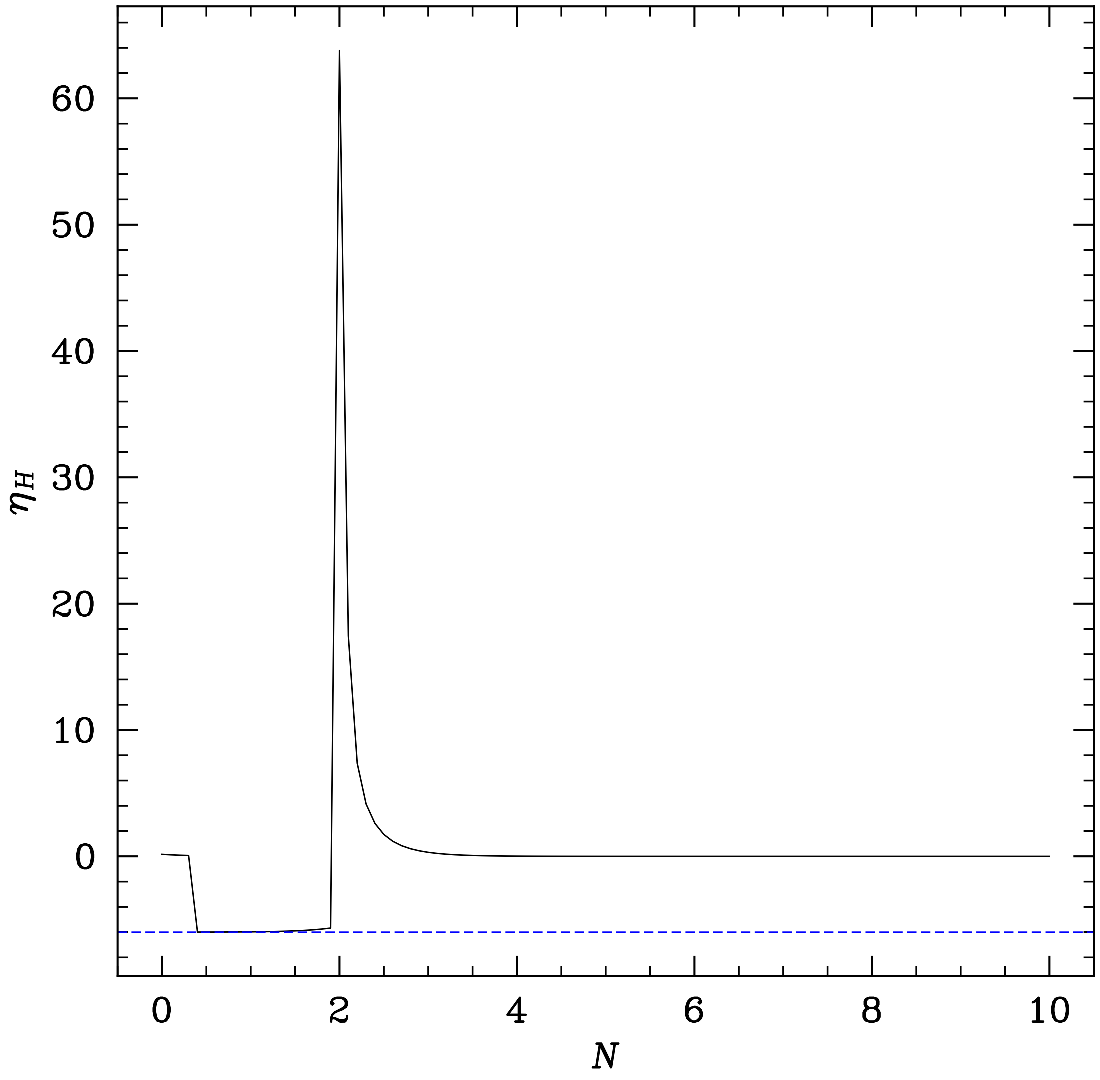


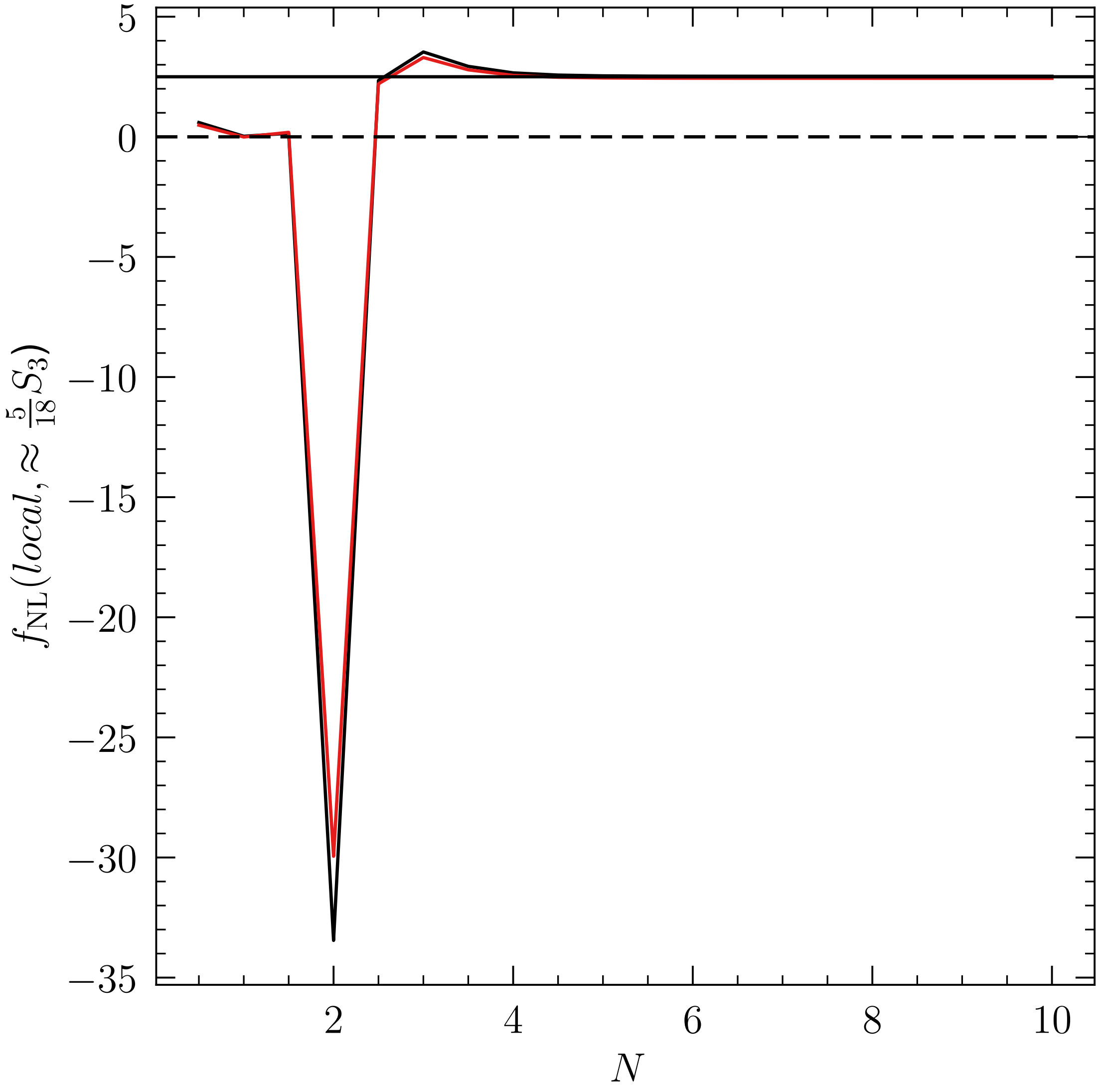
$A_- = A_+/1700,\ H_0 = \sqrt{V_0/(3M_P^2)} = 10^{-5}M_P, L = 10/a_{ini}H_{ini}$

Results: Strobinsky linear potential w/2 steps



Passaglia, Hu, & Motohashi (18),….., Pi & Sasaki (22),….

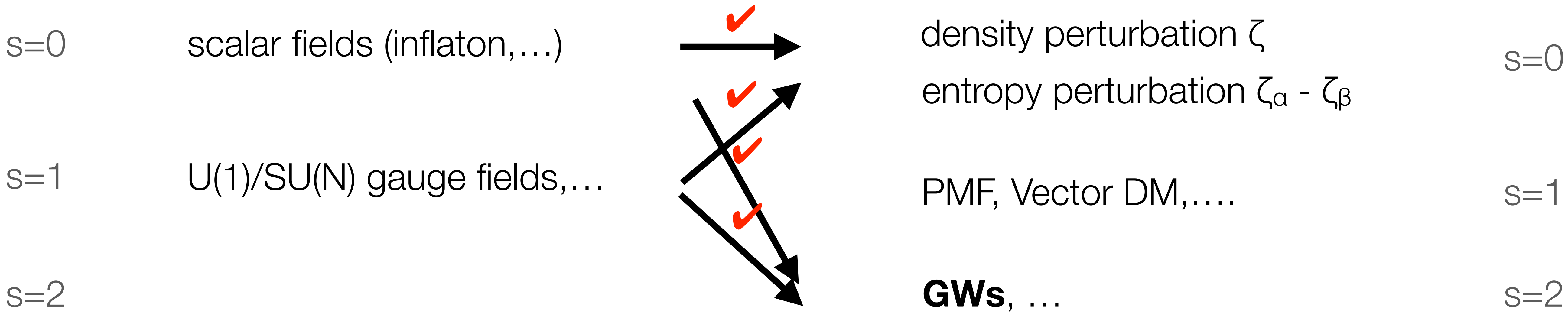




Summary : 2 generalizations

1) Going beyond scalar systems

- From arbitrary (integer) spins to arbitrary spins Tanaka & Y.U. (2021, 2023, 2024)



s=0 scalar fields (inflaton,…) → density perturbation ζ, entropy perturbation ζ_α - ζ_β s=0

s=1 U(1)/SU(N) gauge fields,… PMF, Vector DM,…. s=1

s=2 **GWs**, … s=2

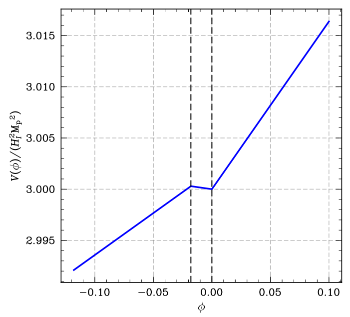
2) Going beyond gradient expansion (k/aH ≦1)

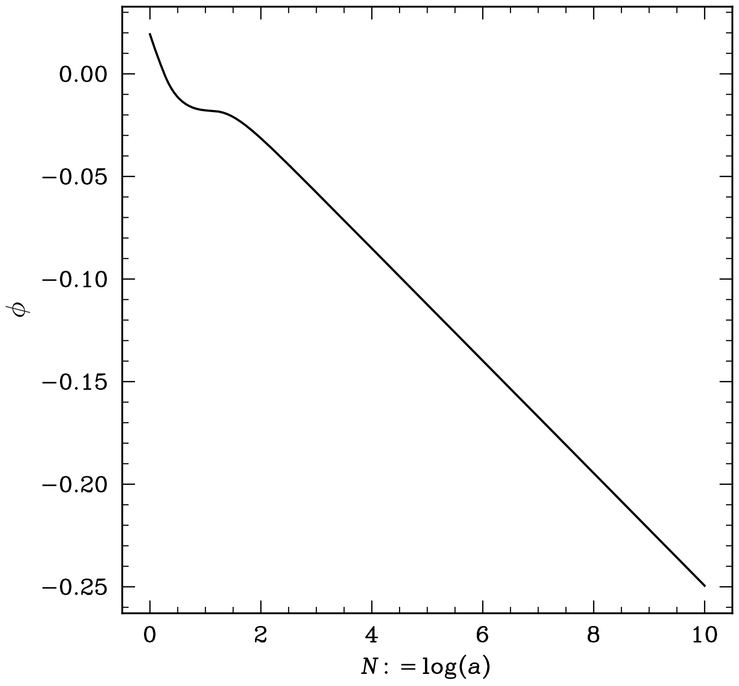
Beyond gradient expansion, Incl. initial NG,… w/ Saha, Tada (in prep)

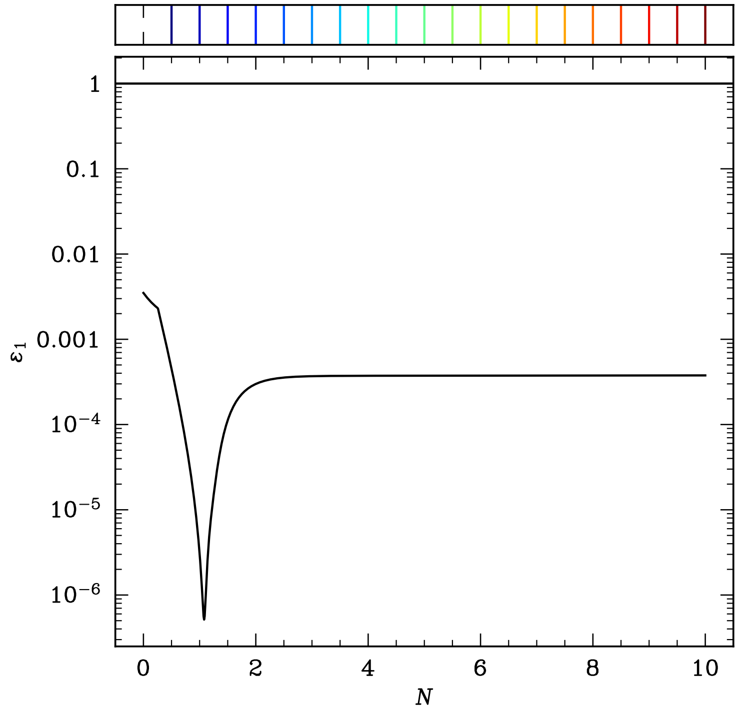
(Metric perturbation is not yet fully included.)

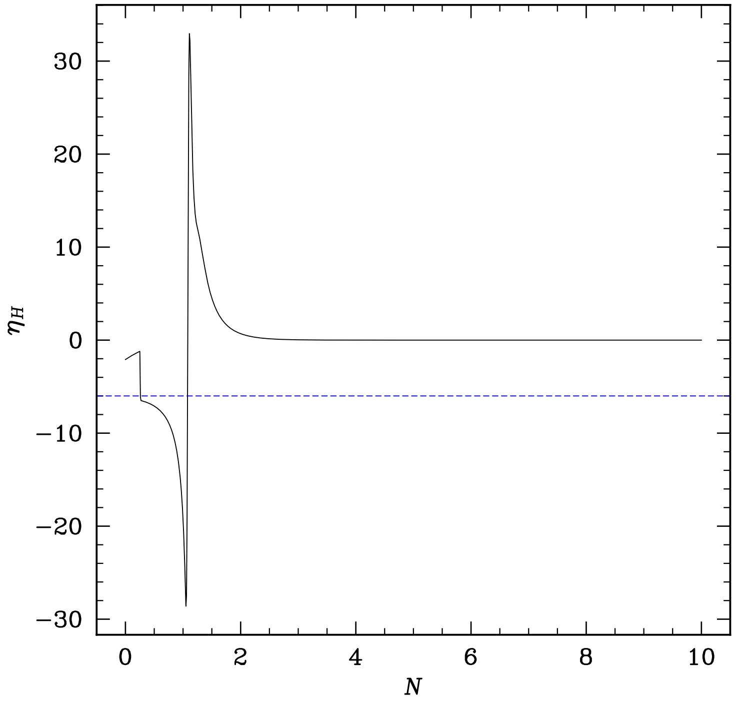
Supplement

Upward step


3.015
3.010
3.005
3.000
2.995
$V(\phi)/(H_I^2 \mathrm{M_p}^2)$
−0.10
−0.05
0.00
0.05
0.10
ϕ


0.00
−0.05
−0.10
−0.15
−0.20
−0.25
ϕ
0
2
4
6
8
10
$N := \log(a)$


1
0.1
0.01
0.001
10^{-4}
10^{-5}
10^{-6}
ε_1
0
2
4
6
8
10
N


30
20
10
0
−10
−20
−30
η_H
0
2
4
6
8
10
N

Current test of isotropy

for test of homogeneity, see analyses in LTB model, e.g., Alnes &Amarzguioui (06), Kolb&Lamb(09), Saito, Ishibashi, & Kodama(10),…

~~Global isotropy~~ in the spectrum of the primordial density perturbation

Imprints on CMB

Soda, Kanno, & Watanabe (10)

$$\mathcal{P}_{\mathcal{R}}(\boldsymbol{k}) = \mathcal{P}^0_{\mathcal{R}}(k)\left[1 + g(k)\left(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{d}}\right)^2\right]$$

➡

- Non-trivial (l, m) dependence

$$\langle a^X_{lm} a^{Y\,*}_{l'm'}\rangle = C^{XY}_{l,l';m}\delta_{m,m'}$$

X, Y = δT/T, E, B

$$C^{XY}_{l,l';m} \to C_l\delta_{ll'}$$

for g=0

- $\langle a^B_{lm} a^{B*}_{l'm'}\rangle \neq 0$ enen w/o PGWs.

Upper bounds on $g_*=g(k_*)$ ~ 10^{-2}

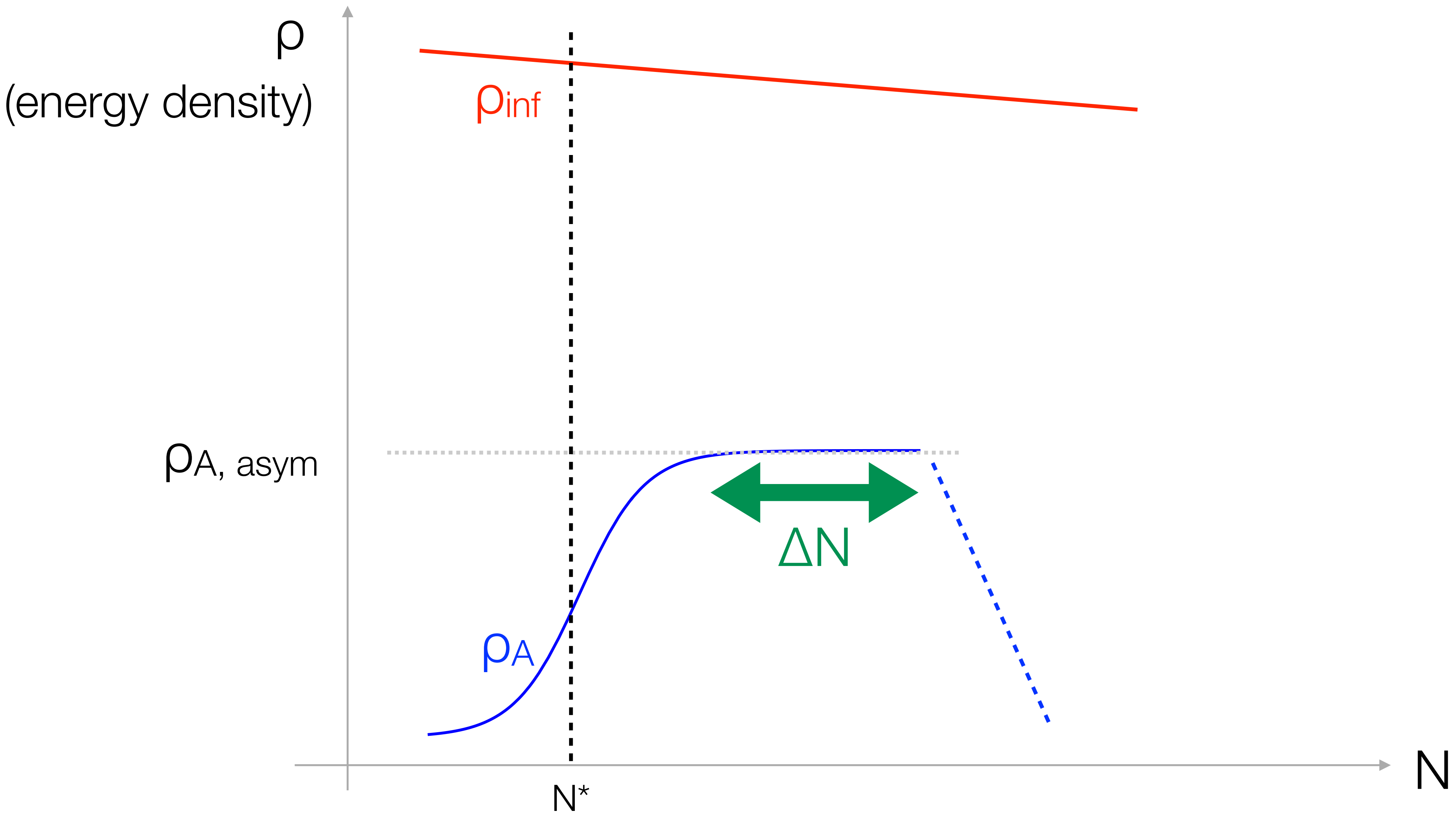
WMAP13 Kim & Komatsu (10) $P(\mathbf{k}) = P_0(k)[1 + g_*(\hat{\mathbf{k}}\cdot\hat{\mathbf{E}}_{\rm cl})^2]$ $g_* = 0.002 \pm 0.016$ (68% CL)

PLANCK18 $\mathcal{P}_{\mathcal{R}}(\boldsymbol{k}) = \mathcal{P}^0_{\mathcal{R}}(k)\left[1 + g(k)\left(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{d}}\right)^2\right]$ $g^* < 0.01$ for q=-2, -1, 0, 1, 2

$$g(k) = g_*(k/k_*)^q,$$

BOSS DR12(LSS) Sugiyama, Shiraishi, Okumura (17) ~ factor 2 weaker than CMB

A benchmark model



Model building is possible w/a sizable coupling generically under slow-roll approx.

e.g. for $\ln f(\phi) = \lambda \frac{\phi}{M_{\rm pl}}$ $\quad \lambda > 1/\sqrt{\varepsilon}$

Tanaka and Urakawa (24)

See also Fujita et al. (18)

Even for $\rho_A/\rho_{\rm inf} << 1$, the backreaction can become important.

LattEfold

LattEfold (Lattice Evolver in e-folds) by Pankaj Saha (KEK)

- Implementation: Written in C++ with OpenMP parallelization, built from scratch to solve field equations in terms of e-folds.
- Straightforward inclusion of additional fields (scalars, U(1), etc…) and the ignored shear (~ computation of GWs).
- Lattice size tested: N= 256^3 on PC and 512^3 on cluster.

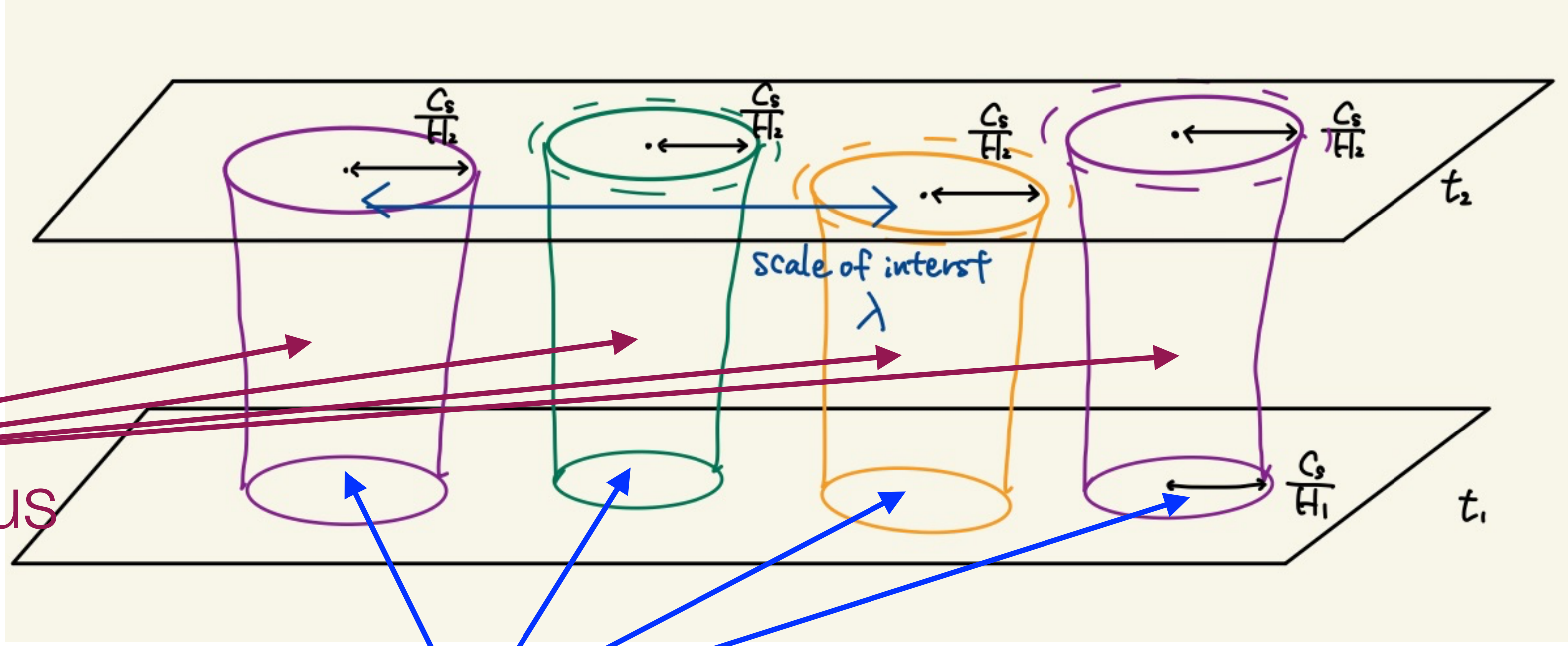
- Lattice directly solves δN.

- Fully non-linear dynamics (incl. non-Gaussianity at horizon crossing)

- Shear, MC yet to be checked

δN formalism

Starobinsky (82, 85),
Sasaki & Stewart (95),
Sasaki & Tanaka (98),
Lyth, Malik, & Sasaki (04),
Lyth & Rodriguez (05),...



reheating surface
w/uniform density

Hubble crossing
of CMB scales

homogeneous
e-folding

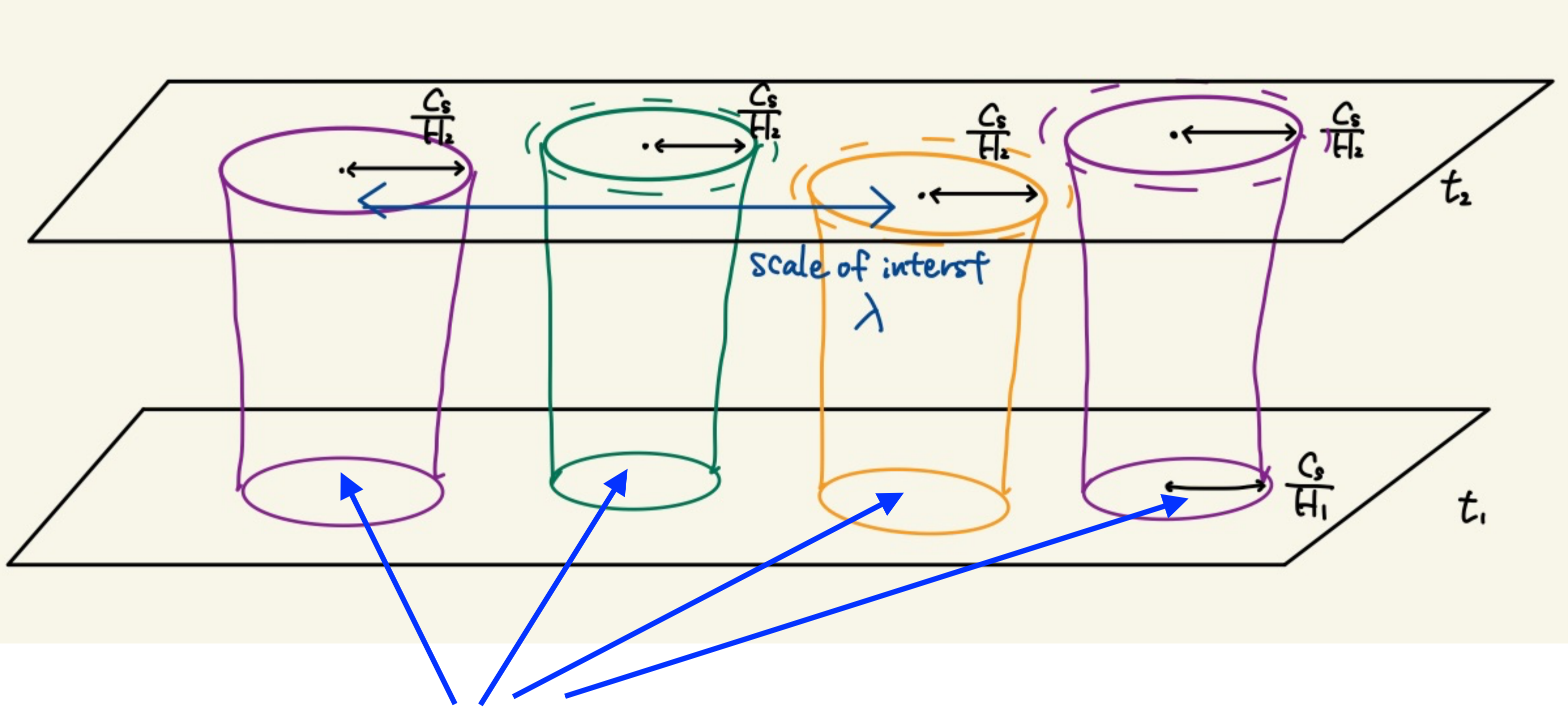
Inhomogeneity is described by different ICs for homogeneous small universes.

1. Compute correlates based on QFT in curved space.
2. Assign IC of (ϕ, Πϕ,…) for each small universe as a random variable with the statistics determined by quantum correlators.
3. Solve the ODEs (classical eoms) for each IC.

→ primordial density perturbation <ζζ>, <ζζζ>,….. (+isocurvature)

generalized δN (gdeltaN) formalism

Tanaka & Y.U. (21)



reheating surface
w/uniform density

Hubble crossing
of CMB scales

Inhomogeneity is described by different ICs for homogeneous small universes.

1. Compute correlates based on QFT in curved space, satisfying all eqs.
2. Assign IC of (ϕ, Π_ϕ, A^i, Π_{A^i}...) for each small universe as a random variable with the statistics determined by quantum correlators, using {φ_{phys}, φ_{auxi}} where φ_{auxi} *can be eliminated by solving MC.*
3. Solve the ODEs w/o MC for each IC.

→ primordial density perturbation $\langle\zeta\zeta\rangle$, $\langle\gamma\gamma\rangle$, $\langle\zeta\zeta\zeta\rangle$, $\langle\gamma\gamma\gamma\rangle$,....

Noether charges in separate Universe

Tanaka & Y.U. (2021)

Infinitesimal transformation $x^i \to \tilde{x}^i = x^i + M^i{}_j(\boldsymbol{x})\, x^j$ $\left|\frac{c_s}{K}\partial_i M^k{}_l\right| \ll |M^k{}_l|$

$$0 = \int d^{d+1}x \Bigg[\underbrace{M^i{}_i N\sqrt{g}\mathcal{L}}_{\text{volume change}} + \mathcal{H}\delta_M N + \frac{\partial(N\sqrt{g}\mathcal{L})}{\partial g_{ij}}\delta_M g_{ij} + \pi^{ij}\delta_M \dot{g}_{ij}$$

$$+ \sum_\alpha \frac{\partial(N\sqrt{g}\mathcal{L})}{\partial \varphi^\alpha_{\mathrm{matter}}}\delta_M \varphi^\alpha_{\mathrm{matter}} + \sum_\alpha \pi^\alpha_{\mathrm{matter}}\delta_M \dot{\varphi}^\alpha_{\mathrm{matter}} \Bigg]$$

e.g. $\delta_M g_{ij} = \tilde{g}_{ij}(t,\, \tilde{x}^i) - g_{ij}(t,\, x^i) = -M^k{}_i g_{kj}(t,\, x^i) - M^k{}_j g_{ik}(t,\, x^i) + \mathcal{O}(M^2,\, \epsilon)$

$$\pi^{ij} \equiv \frac{\partial(N\sqrt{g}\mathcal{L})}{\partial \dot{g}_{ij}} \qquad \pi^\alpha_{\mathrm{matter}} \equiv \frac{\partial(N\sqrt{g}\mathcal{L})}{\partial \dot{\varphi}^\alpha_{\mathrm{matter}}}$$

$$\longrightarrow \quad 0 = \int d^{d+1}x M^i{}_j(\boldsymbol{x}) \left[N\sqrt{g}\mathcal{L}\,\delta^j{}_i + \partial_t \left(-2\pi^j{}_i + \sum_\alpha \pi^\alpha_{\mathrm{matter}} \frac{\partial \delta_M \varphi^\alpha_{\mathrm{matter}}}{\partial M^i{}_j} \right) + \mathcal{O}(\epsilon) \right]$$

$\longrightarrow$ Noether charge (density) $8\pi G \left[-2\pi^j{}_i + \sum_\alpha \pi^\alpha_{\mathrm{matter}} \frac{\partial \delta_M \varphi^\alpha_{\mathrm{matter}}}{\partial M^i{}_j} + \mathcal{O}(\epsilon) \right]^{\mathrm{TL}} \equiv Q^j{}_i(\boldsymbol{x})$

$$\pi^j{}_i \sim \sqrt{g} A^i_j$$

Spatial gauge condition

(d+1)-dim spacetime

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \qquad g_{ij} = e^{2\psi}\gamma_{ij} \qquad \det[\gamma] = 1$$

$$\boxed{N_i = 0 \quad + \quad \partial_j \gamma_{ij}(t_*, \boldsymbol{x}) = 0}$$

Residual gauge

$$\tilde{N}_i = N_i - \delta\dot{x}^j g_{ji} \qquad (\, x^i \to \tilde{x}^i = x^i + \delta x^i \,)$$

Extrinsic curvature $K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - D_i N_j - D_j N_i)$ $\quad K \equiv g^{ij} K_{ij}$

trace: Expansion of volume $K = \frac{d}{N}\dot{\psi}$

traceless: Shear $A^i{}_j = \frac{1}{2N}\gamma^{im}\dot{\gamma}_{mj}$

when $A^i{}_j \propto 1/V_{\rm phys} \sim 1/a^d$, $\partial_j \gamma_{ji}(t, \mathbf{x}) = 0, \; \psi \to \zeta$ at linear.

Non-trivial example

$$\mathcal{L}(x) = \cdots + g^{ij}\partial_i\chi\partial_j\chi/2 + \chi\mathcal{F}_\phi(\phi(x)) + \cdots \ ,$$

χ: Lagrange multiplier

$$\longrightarrow \quad \frac{1}{N\sqrt{g}}\partial_i(g^{ij}N\sqrt{g}\partial_j\chi(x)) = \mathcal{F}_\phi(\phi(x))$$

Solving constraints gives rise to non-local contributions

$\mathcal{H}_i - \partial_i \mathcal{H} = (\cdots)/V_{\rm phys}$

Garriga, Y.U., & Vernizzi(16)

GR $\quad S = \int \mathrm{d}^{d+1}x\sqrt{-g}\,P(X^{IJ},\phi^K) \qquad X^{IJ} \equiv -(\partial_\mu\phi^I\partial^\mu\phi^J)/2$

$$K^2 = \frac{2\kappa^2}{d(d-1)}\left(P_{IJ}K^2\partial_{\mathcal{N}}\phi^I\partial_{\mathcal{N}}\phi^J - d^2P\right) + \mathcal{O}(\epsilon^2)\,, \qquad \partial_{\mathcal{N}} \equiv \frac{d}{K\alpha}\partial_t$$

$$\partial_{\mathcal{N}}K = -\frac{\kappa^2}{d-1}KP_{IJ}\partial_{\mathcal{N}}\phi^I\partial_{\mathcal{N}}\phi^J + \mathcal{O}(\epsilon^2)\,,$$

$$K\partial_{\mathcal{N}}\left(P_{IJ}K\partial_{\mathcal{N}}\phi^J\right) + dK^2P_{IJ}\partial_{\mathcal{N}}\phi^J - d^2(\partial P/\partial\phi^I) = \mathcal{O}(\epsilon^2)$$

MC $\quad \partial_iK = -\frac{\kappa^2}{d-1}KP_{IJ}\partial_{\mathcal{N}}\phi^I\partial_i\phi^J + \mathcal{O}(a\epsilon^3)$

∂iHC $\quad \partial_iK = -\frac{\kappa^2}{d-1}KP_{IJ}\partial_{\mathcal{N}}\phi^I\partial_i\phi^J + B_i + \mathcal{O}(a\epsilon^3)$

$$B_i \equiv \frac{\kappa^2K}{(d-1)\partial_{\mathcal{N}}\ln(e^{d\mathcal{N}}K)}\left[\partial_{\mathcal{N}}\phi^I\partial_i\left(P_{IJ}\partial_{\mathcal{N}}\phi^J\right) - \partial_{\mathcal{N}}\left(P_{IJ}\partial_{\mathcal{N}}\phi^J\right)\partial_i\phi^I\right] \qquad a^{-1}B_i = O(\epsilon^3)$$

Shear as $\mathcal{O}(\epsilon^0)$

Tanaka & Y.U. (2021)

GR + $\mathcal{L}_{\mathrm{matter}} = P(X,\,\phi)$ $\quad X \equiv -\partial_\mu\phi\partial^\mu\phi$ $\quad$ spatial gauge w/N_i=0

HC $$\frac{1-d}{d}K^2 + A^i_j A^j_i + 16\pi G\rho = \mathcal{O}(\epsilon)$$

MC $$\left(\frac{1}{d}-1\right)\partial_i K + \nabla_j A^j{}_i - 16\pi G P_X \frac{\dot{\phi}}{N}\partial_i\phi = \mathcal{O}(\epsilon^2)$$

Noether $$A^i{}_j = -Q^i{}_j/\sqrt{g}$$

at linear perturbation, δϕ=0 $\quad A^i{}_j \sim \frac{1}{2}\delta^{im}\dot{\gamma}_{mj} \sim \frac{1}{2}\delta^{im}\left(\partial_m\partial_j\partial^{-2} - \frac{1}{d}\delta_{mj}\right)\dot{\chi}$

$$\longrightarrow \quad \delta\dot{\psi} = \frac{1}{4d}\left\{d(1-d) + \frac{2}{c_s^2}\right\}\frac{c_s^2}{\varepsilon}\dot{\chi} + \mathcal{O}(\epsilon)$$

$\dot{\chi} \propto a^{-d}$

$\varepsilon \equiv -\dot{H}/H^2$

$\longrightarrow$ if $A^i{}_j = \mathcal{O}(\epsilon)$ $\qquad \delta\dot{\psi} = \mathcal{O}(\epsilon)$

* FLRW limit is not $\epsilon \to 0$ but $\epsilon \to 0$ + late time limit.