



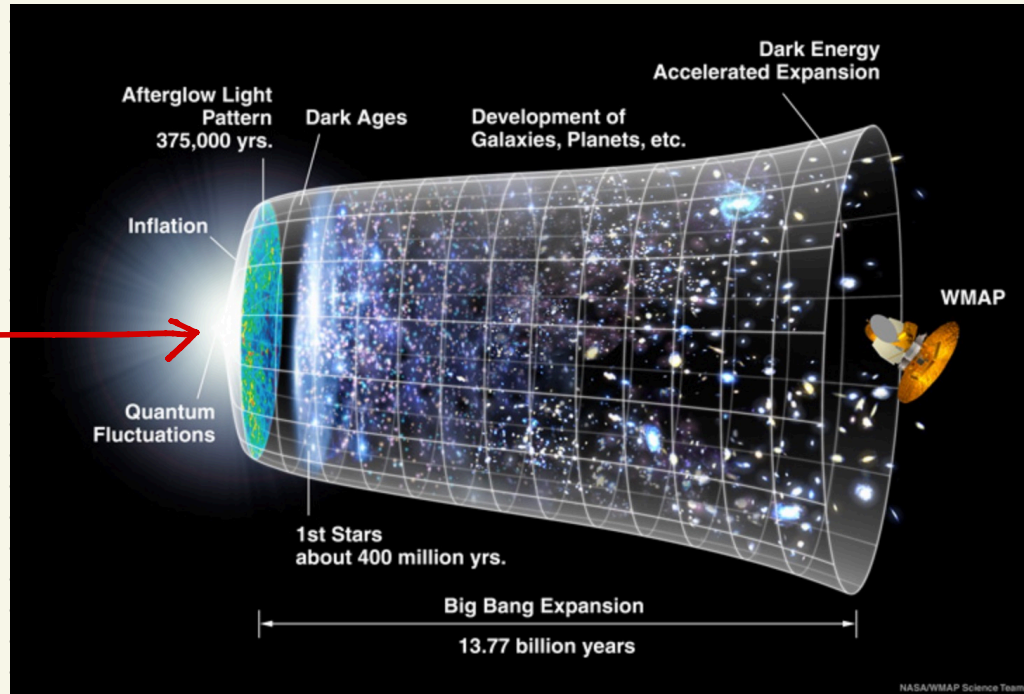
# Large, rare fluctuations from Stochastic Inflation

David Wands

Jackson, Assadullahi, Gow, Koyama, Vennin & Wands, arXiv: 2311.03281, 2410.13683

Artigas, Frion, Miranda, Vennin & Wands, arXiv: 2504.05937

inflation provides initial conditions for cosmology



large, homogeneous, isotropic, spatially-flat FLRW background ✓  
+ primordial perturbations ✓ ❤️

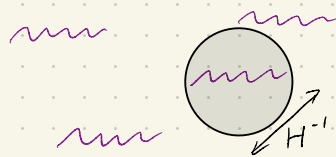
inflation

accelerated expansion ( $\ddot{a} > 0$ ) in v.e.u.

e.g. quasi-de Sitter :  $H = \frac{\dot{a}}{a} \approx \text{constant}$

quantum vacuum

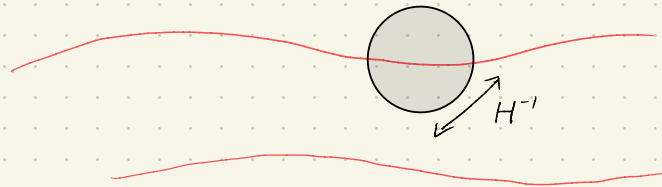
UV ( $k \gg aH$ )



inflation  
→

primordial density fluctuations

IR ( $k \ll aH$ )



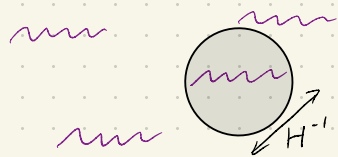
inflation

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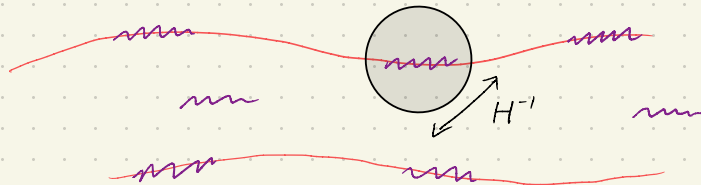
UV ( $k \gg aH$ )



inflation  
 $\longrightarrow$

primordial density fluctuations

IR ( $k \ll aH$ )



sufficient inflation ( $N = \int H dt > 70$ )  
generates density fluctuations on all scales



# classical vs quantum inflation

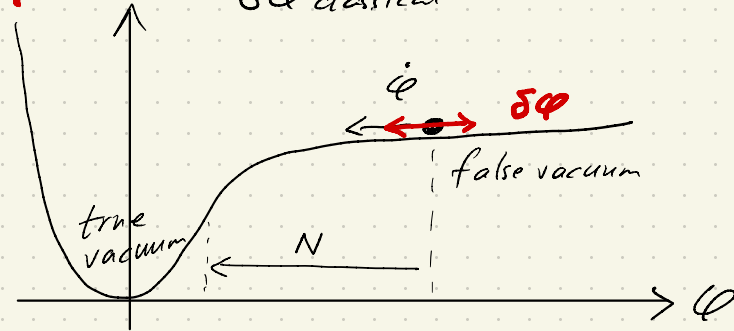
- classical slow-roll inflates universe

$$\rightarrow \ln(a) = N = \int H dt = \int \frac{H}{\dot{\phi}} d\phi$$

- quantum field fluctuations perturb local adiabatic expansion

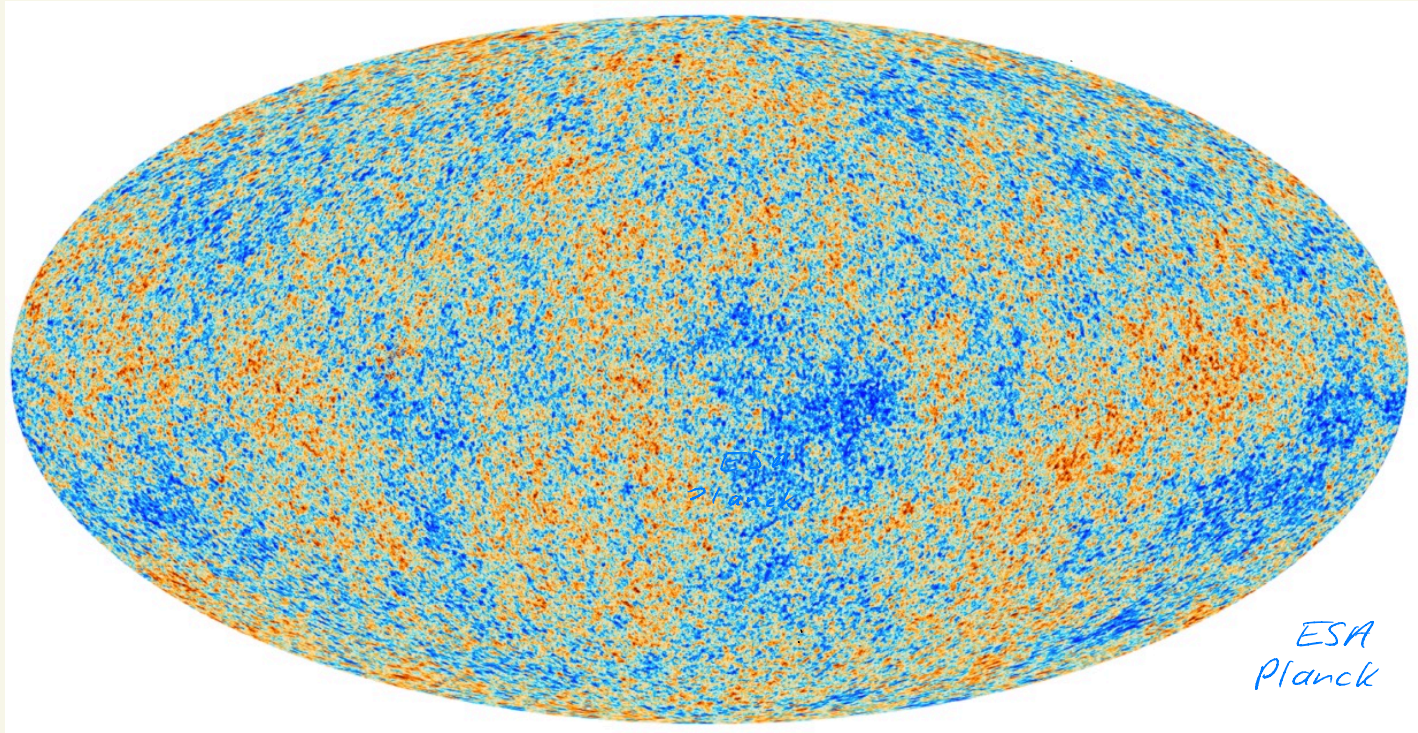
$\rightarrow$  adiabatic curvature perturbation

$$R = \delta N = H \frac{\delta \phi}{\dot{\phi}} = \frac{\delta \mathcal{L}_{\text{quantum}}}{\delta \mathcal{L}_{\text{classical}}}$$



# primordial density perturbations from inflation

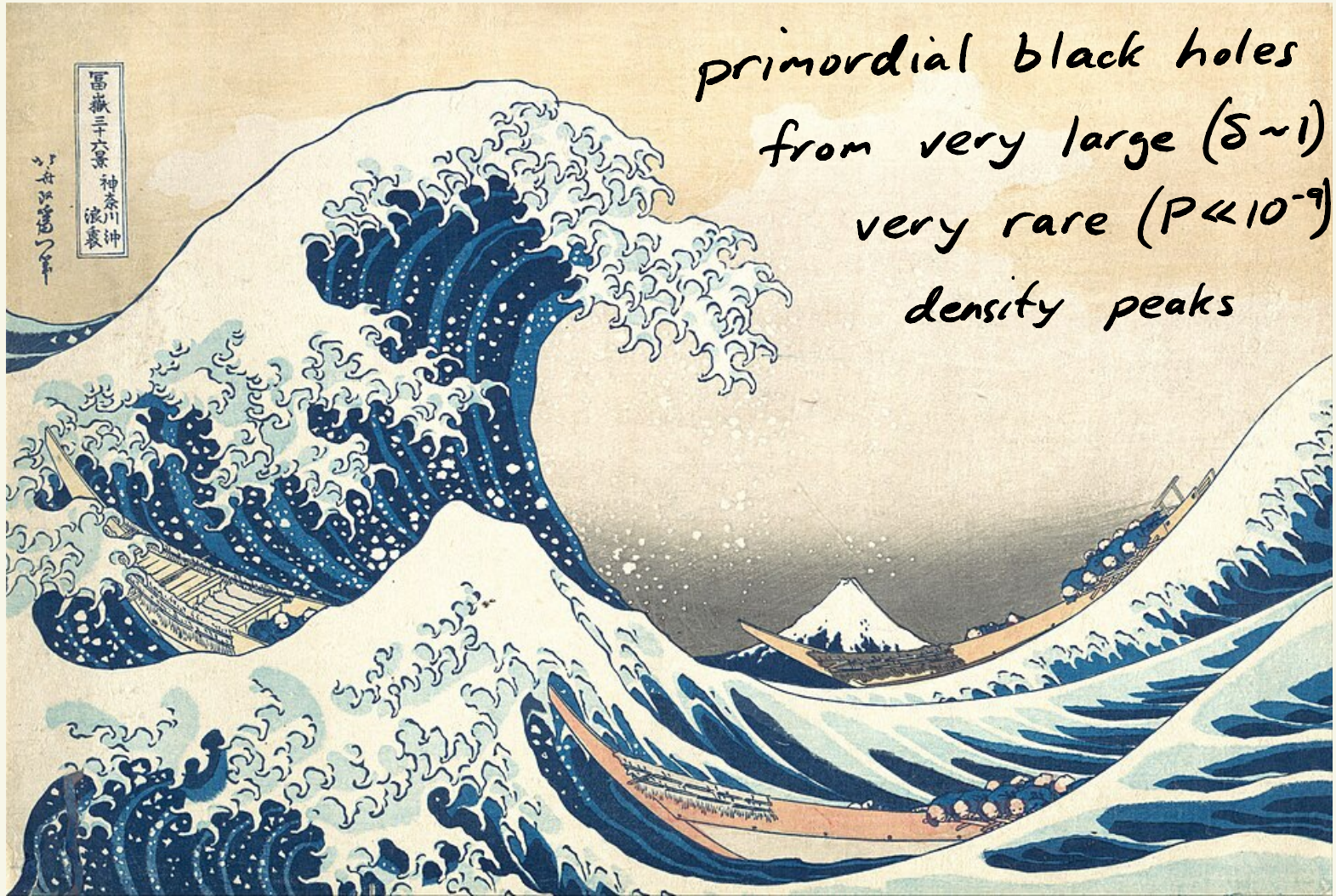
$$\frac{\Delta T}{T} \approx \frac{2}{5} R \sim 10^{-5} \quad \text{adiabatic, Gaussian, } \approx \text{scale invariant}$$



ESA  
Planck



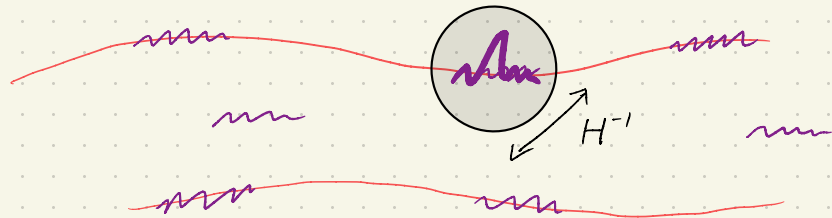
primordial black holes  
from very large ( $\delta \sim 1$ )  
very rare ( $P \ll 10^{-9}$ )  
density peaks



# Primordial black holes

Zeldovich & Novikov (1967), Hawking (1971)  
Hawking & Carr (1974)

- formed by collapse of large density perturbations,  $\frac{\delta \rho}{\rho} > \delta_{crit}$   
in early radiation-dominated era
- mass related to horizon mass at formation time



$$H^{-1} = 2t$$

$$M_{PBH} \sim \frac{t}{10^{-6} \text{ s}} M_{\odot}$$

- non-perturbative, large,  $\delta \sim 1$ , and very rare

mass fraction at formation :  $\beta(M) \sim 10^{-8} \left( \frac{M_{PBH}}{M_{\odot}} \right)^{1/2} \Omega_{PBH}$

# ultra-slow-roll inflation

e.g., Garcia-Bellido & Ruiz (2017)

primordial curvature  
power spectrum

$$\langle R^2 \rangle = \left( \frac{H}{\dot{\varphi}} \right)^2 \langle \delta\varphi^2 \rangle$$

↻ boosted as  $\left( \frac{\dot{\varphi}}{H} \right)^2 \rightarrow 0$

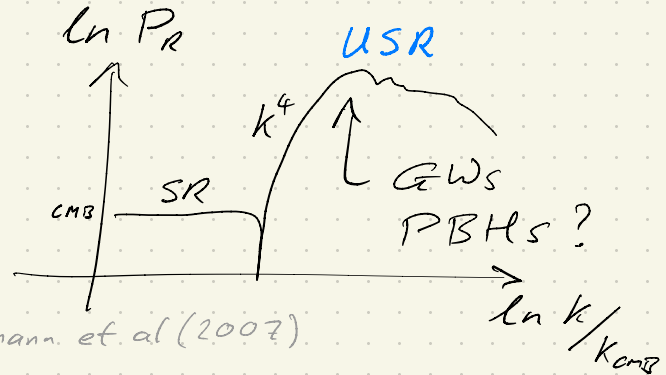
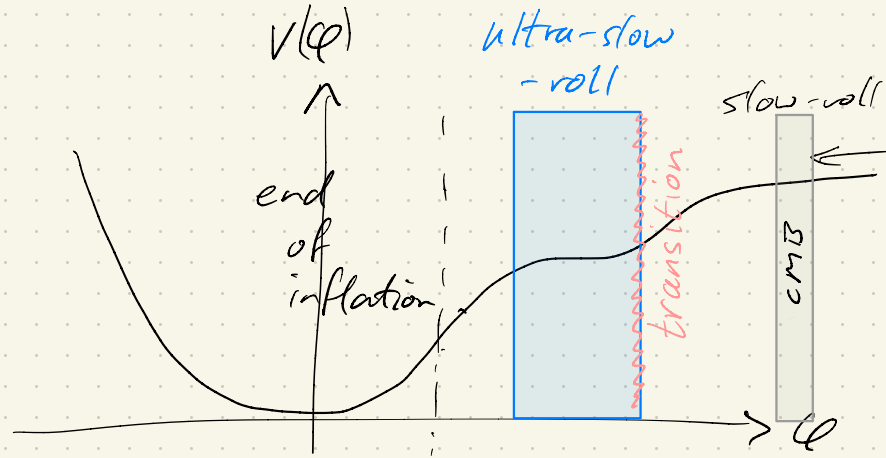
see talks by Caravano & Pi

\* **primordial black holes**

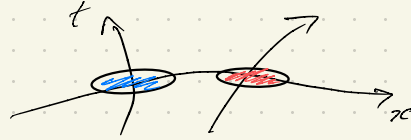
Carr & Hawking '74

\* **induced grav. waves**

Matarrese et al., Ananda et al., Baumann et al (2007)



# scalar field in separate FRW universes



$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0$$

two regimes

(i) quasi-equilibrium :  
 $|\ddot{\phi}| \ll |3H\dot{\phi}|$

$$\dot{\phi} \simeq -\frac{V'}{3H} \rightarrow \text{slow-roll attractor, } \dot{\phi}(\phi)$$

(ii) free evolution :  
 $|V'| \ll |3H\dot{\phi}|$

$$\ddot{\phi} \simeq -3H\dot{\phi} \rightarrow \text{ultra-slow-roll}$$

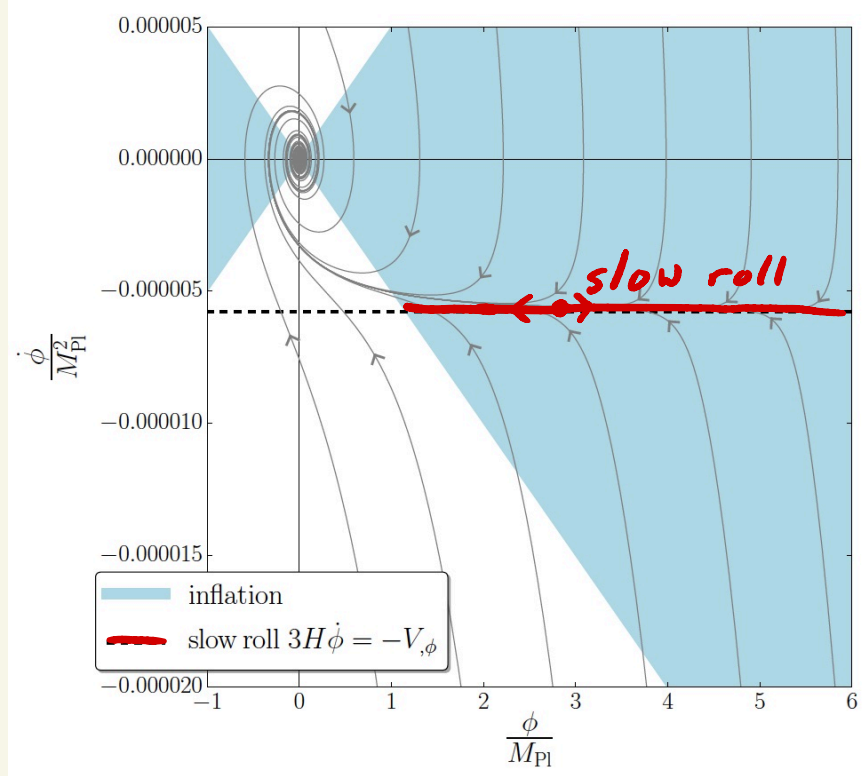
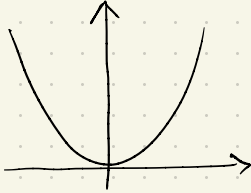
$$\phi \simeq \mathcal{D} \int \frac{dt}{a^3}$$

transient/decaying solution  
as  $a \rightarrow \infty$

# single-field inflation phase space

Grain & Vennin (2017)

$$V = \frac{1}{2} m^2 \phi^2$$



adiabatic expansion

$$N = \int H dt$$

adiabatic perturbation

$$R = -\delta N \\ = H \frac{\delta \phi}{\dot{\phi}}$$

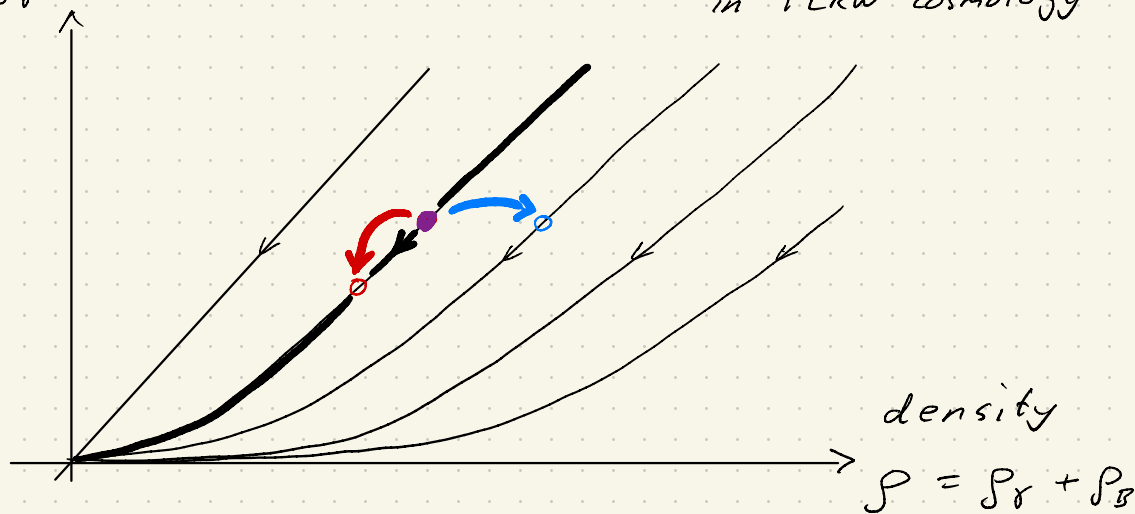
$R = \text{constant}$  on super-Hubble scales ( $k < aH$ )

pressure

$$P = \frac{1}{3} P_\gamma$$

## photon - baryon phase space

in FLRW cosmology



**adiabatic** perturbation

$$R = H \frac{\delta P}{\dot{P}} = H \frac{\delta P}{\dot{P}}$$

$$\Rightarrow \delta \left( \frac{n_\gamma}{n_B} \right) = 0$$

**non-adiabatic** perturbation

$$S = 3H \left( \frac{\delta P}{\dot{P}} - \frac{\delta P_B}{\dot{P}_B} \right)$$

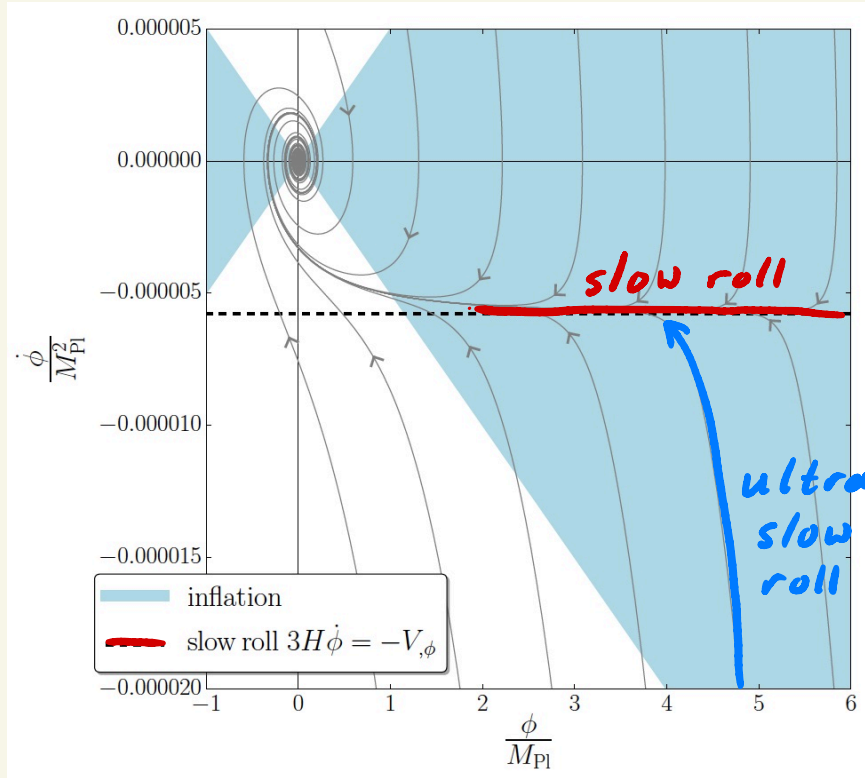
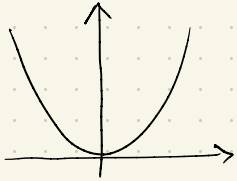
a.k.a. isocurvature,  
entropy perturbation.



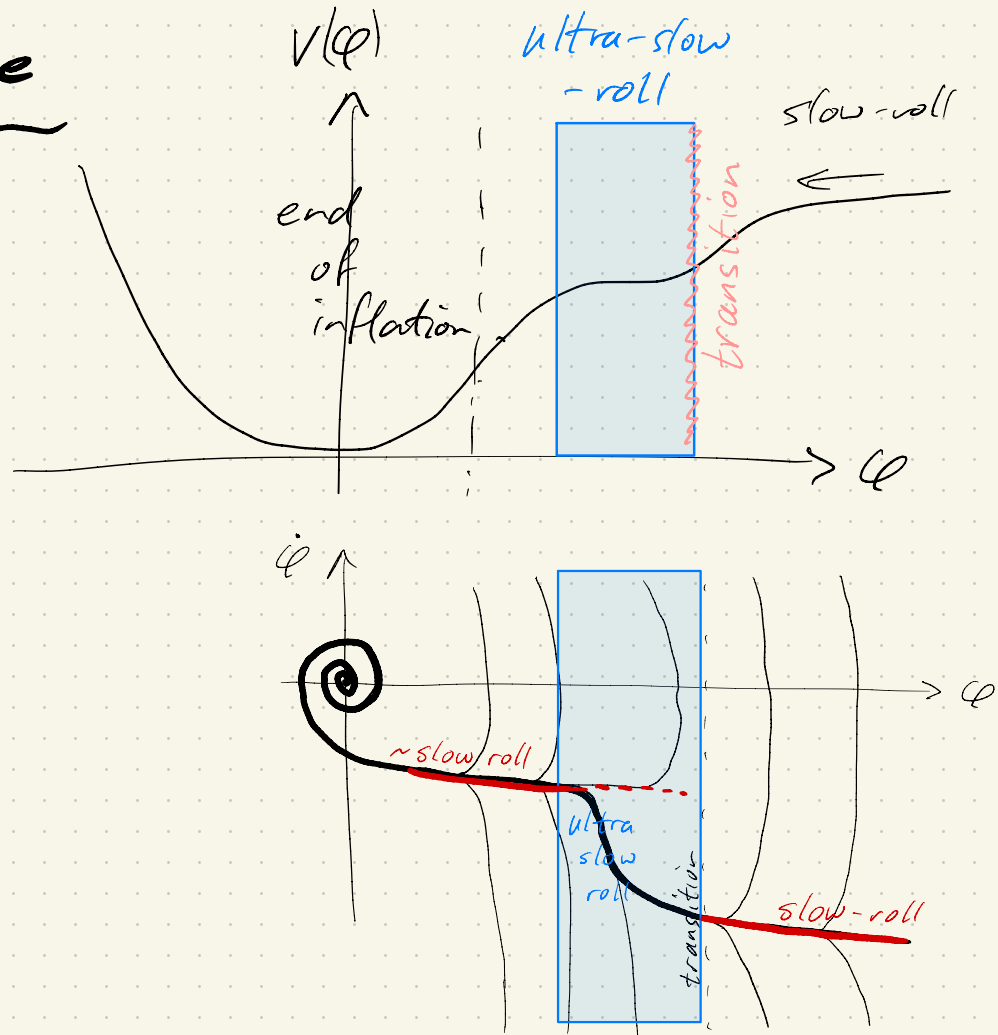
# single-field inflation phase space

Grain & Vennin (2017)

$$V = \frac{1}{2} m^2 \varphi^2$$



# USR phase space



# non-adiabatic pressure in single-field inflation

curvature evolution :  $\dot{R} = -\frac{3H^2}{2\dot{V}} \delta P_{nad}$

FLRW cosmology :  $\rho = \frac{1}{2} \dot{\varphi}^2 + V(\varphi)$  ,  $P = \frac{1}{2} \dot{\varphi}^2 - V(\varphi)$

non-adiabatic perturbation :  $\delta P_{nad} = \delta P - (\dot{P}/\dot{\rho}) \delta \rho$

$= 0$  if  $\exists$  attractor  $\dot{\varphi} = \dot{\varphi}(\varphi)$   
on large scales **in slow roll**

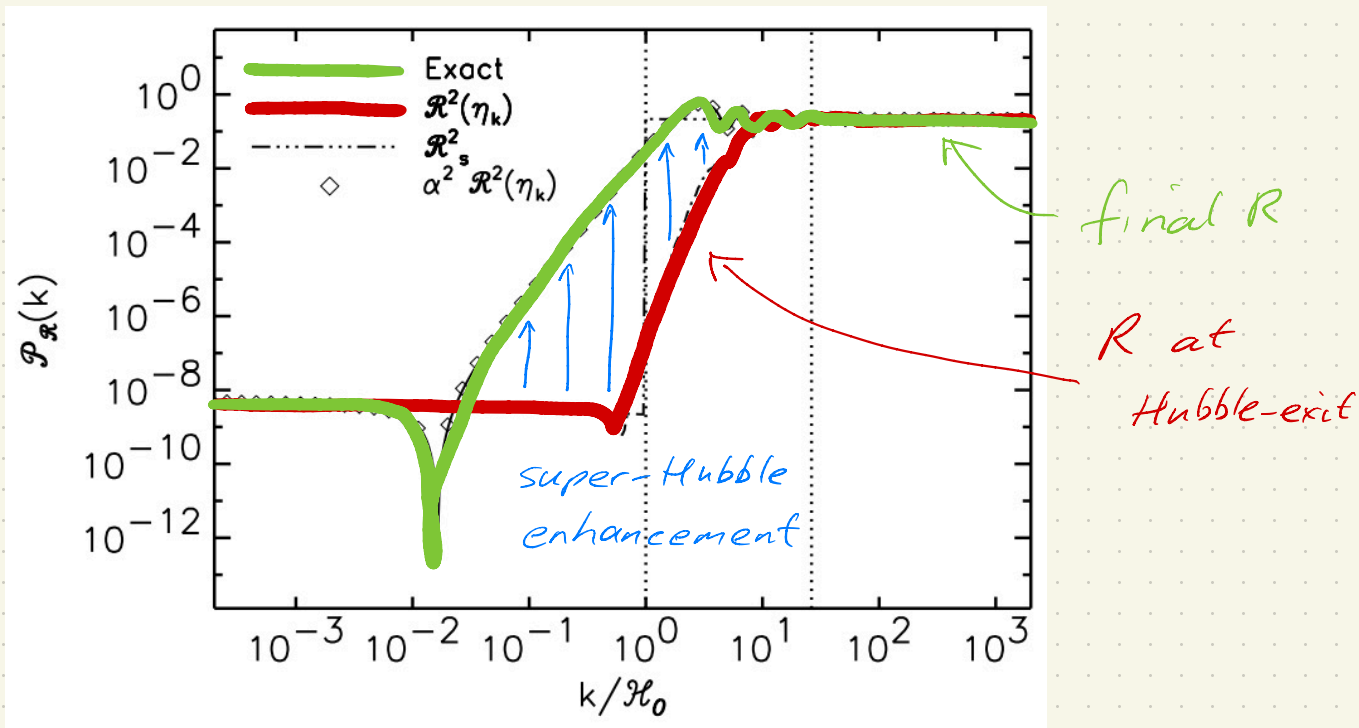
$\neq 0$  on large scales

in ultra slow roll

$$\dot{\varphi} \neq \dot{\varphi}(\varphi)$$

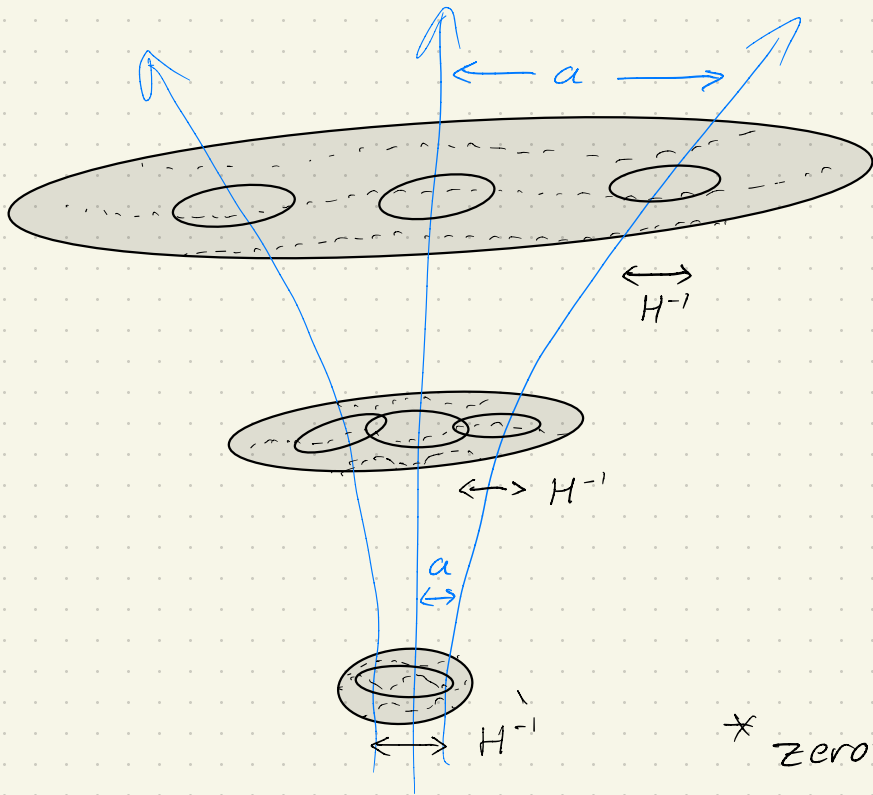
# Enhancement of superhorizon scale inflationary curvature perturbations

Samuel M. Leach<sup>1</sup>, Misao Sasaki<sup>2</sup>, David Wands<sup>3</sup> and Andrew R. Liddle<sup>1</sup>



# Separate universes

see talk by Yuko Urakawa



coarse-grained  $\tilde{\varphi}(\vec{x}) + \tilde{H}(\vec{x})$

obey **nonlinear FLRW** eqns

if  $\neq$  anisotropy and  
spatial gradients negligible\*  
on large scales

\* zeroth-order in gradient expansion

Salopek & Bond (1990)

# Einstein equations in gradient expansion

extrinsic curvature of 3-space :  $K^i_j = -\tilde{H} \delta^i_j + A^i_j$

$$G^0_0 \rightarrow 3\tilde{H}^2 = 8\pi G \left( V(\tilde{\varphi}) + \frac{1}{2} \dot{\tilde{\varphi}}^2 \right) + \mathcal{O}(\nabla^2)$$

$$G^i_i \rightarrow \ddot{\tilde{H}} = -4\pi G \dot{\tilde{\varphi}}^2 + \mathcal{O}(\nabla^2)$$

$$G^i_{j \neq i} \rightarrow \dot{A}^i_j = -3H A^i_j + \mathcal{O}(\nabla^2)$$

$$G^0_i \rightarrow \nabla_i \tilde{H} + 4\pi G \dot{\tilde{\varphi}} \nabla_i \tilde{\varphi} \propto \nabla_j A^j_i + \mathcal{O}(\nabla^3)$$



## Salopek & Bond (1990)

anisotropic evolution eqn:  $\dot{A}_{ij}^i = -3H A_{ij}^i$

$$\Rightarrow A_{ij}^i \propto e^{-3 \int H dt}$$
$$\rightarrow 0$$

momentum constraint neglecting anisotropy

$$\delta H + 4\pi G \dot{\varphi} \delta \varphi = \delta H - \frac{\dot{H}}{\dot{\varphi}} \delta \varphi = 0$$

adiabatic perturbations  $\Rightarrow \tilde{H} = H(\tilde{\varphi})$

Hamilton-Jacobi solution for perturbations

# Artigas et al (2025)

## momentum constraint

see also Tanaka & Urakawa (2021)  
Tomberg (2023)

$$\delta H + 4\pi G \dot{\varphi} \delta \varphi = \delta H - \frac{\dot{H}}{\dot{\varphi}} \delta \varphi \propto e^{-3 \int H dt}$$

dynamically decays but not gradient suppressed

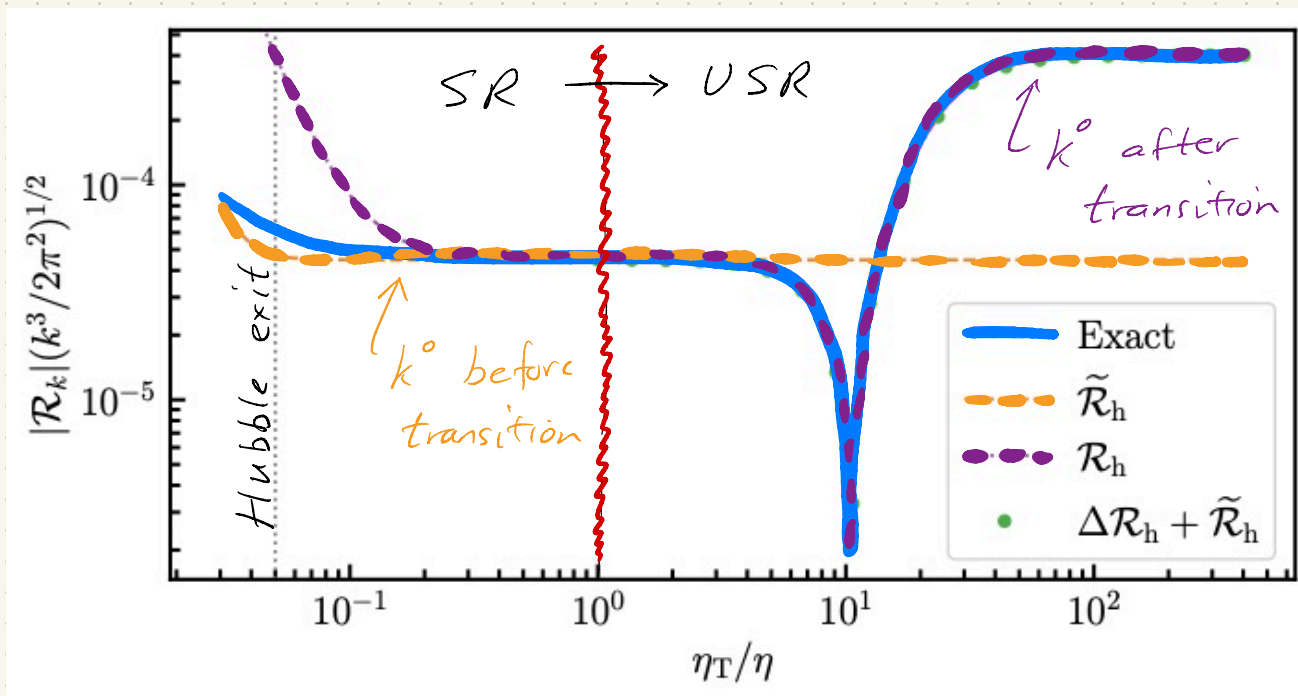
related to non-adiabatic pressure

$$\delta P_{\text{nad}} \propto \frac{v'}{\dot{\varphi}} e^{-3 \int H dt}$$

- e.g.
- slow roll  $\dot{\varphi} \approx \text{constant} \Rightarrow \delta P_{\text{nad}} \propto e^{-3 \int H dt} \rightarrow 0$   
 $\rightarrow$  Hamilton-Jacobi,  $\tilde{H} = H(\tilde{\varphi})$
  - ultra-slow roll  $\dot{\varphi} \propto e^{-3 \int H dt} \Rightarrow \delta P_{\text{nad}} \rightarrow \text{const.}$   
 $\Rightarrow \tilde{H} = H(\tilde{\varphi}, \dot{\tilde{\varphi}})$  2D phase space

# The separate-universe approach and sudden transitions during inflation

Joseph H. P. Jackson,<sup>a</sup> Hooshyar Assadullahi,<sup>a,b</sup> Andrew D. Gow,<sup>a</sup> Kazuya Koyama,<sup>a</sup> Vincent Vennin,<sup>c,a</sup> David Wands<sup>a</sup>

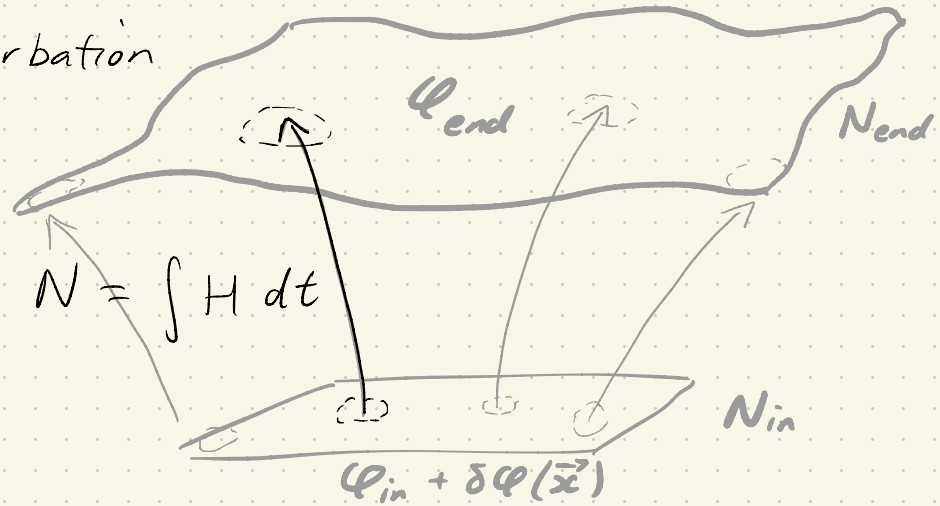


# non-perturbative approach: $\delta N$ formalism

primordial density perturbation

= perturbed expansion

$$\delta N(\vec{x}) = N(\vec{x}) - \bar{N}$$

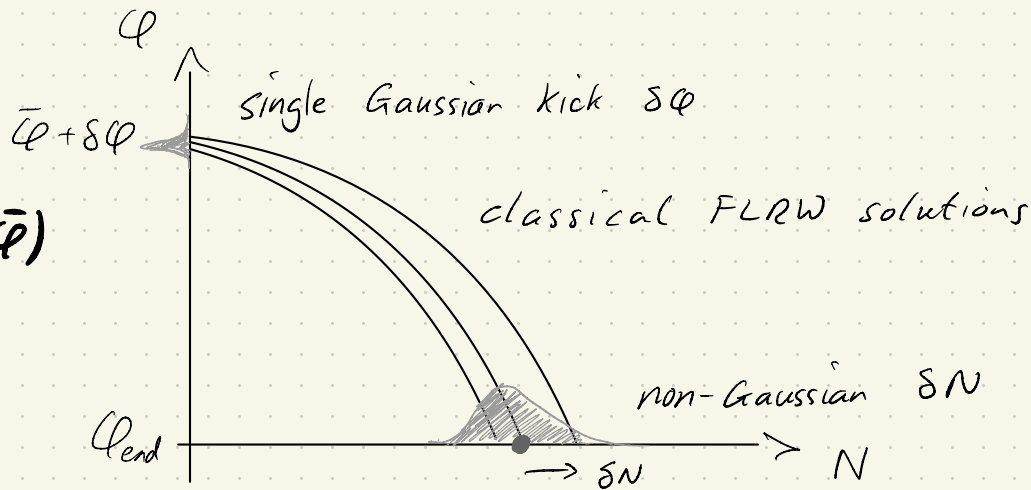


use non-linear FLRW equations on super-Hubble scales  
"separate universe approach"

## Classical $\delta N$

$$\delta N = N(\bar{\varphi} + \delta\varphi) - N(\bar{\varphi})$$

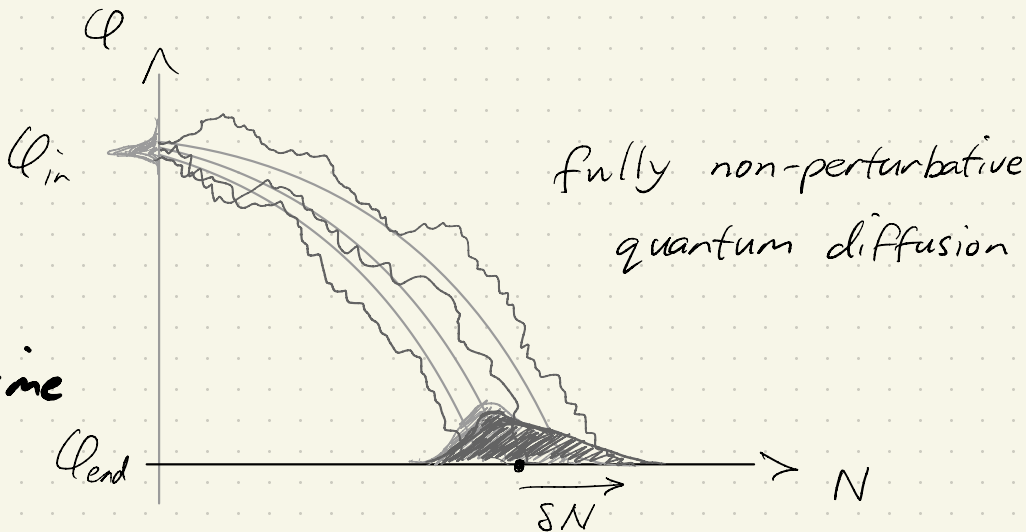
nonlinear classical  
solution  $N(\varphi)$



## Stochastic $\delta N$

$$\delta N = N - \langle N \rangle$$

$N$  = stochastic  
first passage time  
to cross  $\varphi_{\text{end}}$



classical USR

take limit  $V' \rightarrow 0$

$$\ddot{\varphi} + 3H\dot{\varphi} \approx 0 \Rightarrow \dot{\varphi} \propto e^{-3Ht}$$

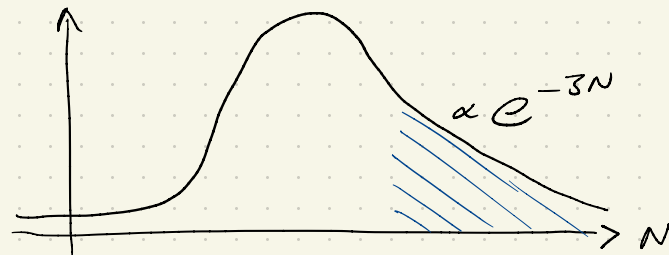
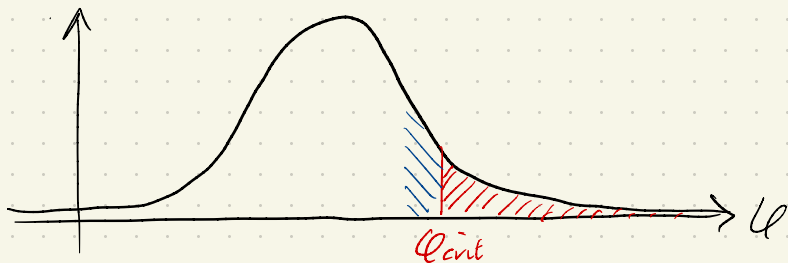
$$N(\varphi, \dot{\varphi}) = -\frac{1}{3} \ln \left( 1 + \frac{3H(\varphi - \varphi_{\text{end}})}{\dot{\varphi}} \right)$$

Cai et al 2017

Pi & Sasaki 2022

$$N \rightarrow \infty \quad \text{as} \quad \varphi \rightarrow \varphi_{\text{crit}} = \varphi_{\text{end}} - \frac{\dot{\varphi}}{3H}$$

Gaussian  $P(\varphi)$   $\longrightarrow$  exponential tail,  $P(N) \propto e^{-3N}$



$\varphi$  trapped if  $\varphi - \varphi_{\text{end}} > -\dot{\varphi}/3H$  (incomplete)

# stochastic USR

take limit  $V' \rightarrow 0$

$$\ddot{\varphi} + 3H\dot{\varphi} \approx 0 \Rightarrow \dot{\varphi} \propto e^{-3Ht}$$

$$N(\varphi, \dot{\varphi}) = -\frac{1}{3} \ln \left( 1 + \frac{3H(\varphi - \varphi_{\text{end}})}{\dot{\varphi}} \right)$$

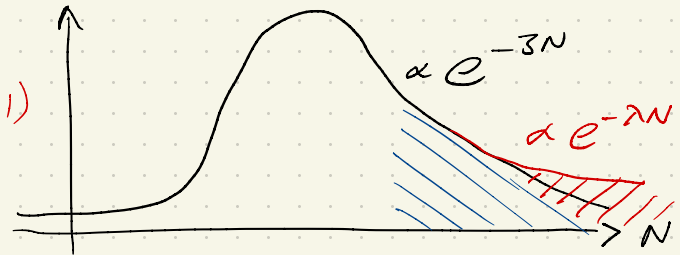
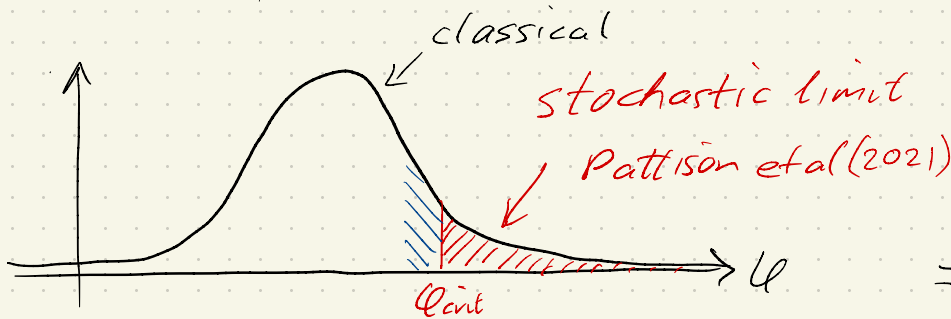
Cai et al 2017

Pi & Sasaki 2022

$$N \rightarrow \infty \quad \text{as} \quad \varphi \rightarrow \varphi_{\text{crit}} = \varphi_{\text{end}} - \frac{\dot{\varphi}}{3H}$$

Gaussian  $P(\varphi)$

exponential tails,  $P(N) \propto e^{-3N}$



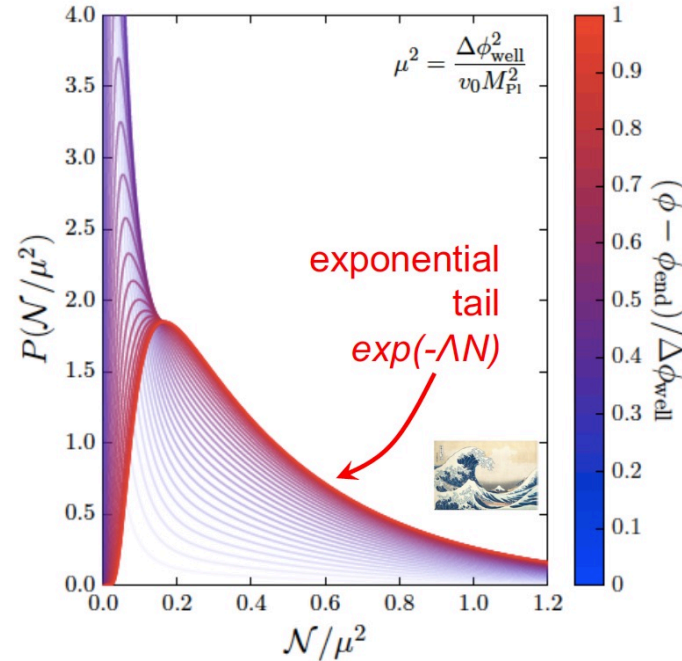
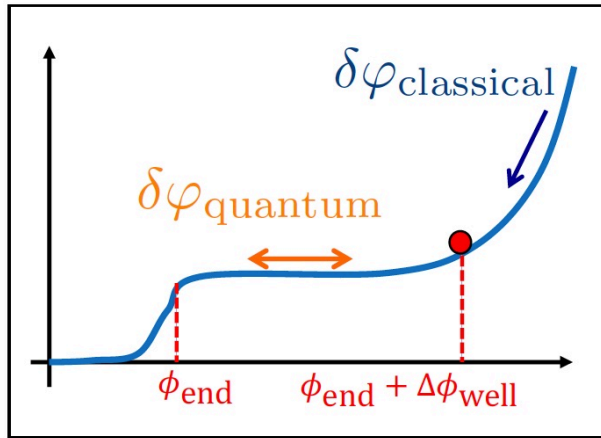
diffusion takes over when  $\varphi - \varphi_{\text{end}} > -\dot{\varphi}/3H$

# Full probability distribution function

Pattison, Vennin, Assadullahi & Wands (2017)



**Stochastic diffusion leads to highly non-Gaussian pdf,  $P(N)$ :**



*probability distribution in general slow-roll has an exponential tail*

Ezquiaga, Garcia-Bellido & Vennin (2019)



# Stochastic Ultra-Slow-Roll Inflation

Fironzjahi et al, Pattison et al (2021)

UV : linear mode functions on sub-Hubble scales

stochastic noise\* as modes cross coarse-graining scale

IR : coarse-grained field obeys Langevin equations  
for  $\varphi$  and  $\pi = a^3 \dot{\varphi}$

$$\frac{d\varphi}{dN} = \frac{\pi}{a^3 H} + \hat{\xi}_{\varphi}$$

$$\frac{d\pi}{dN} = -\frac{a^3 V'}{H} + \hat{\xi}_{\pi}$$

\* squeezed state

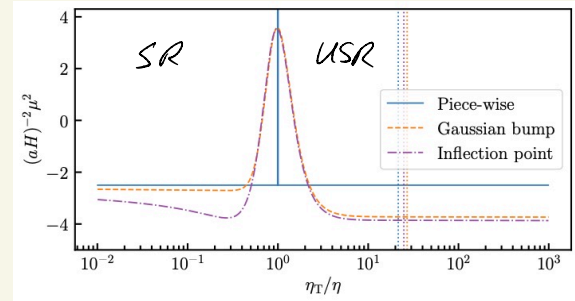
→ correlated noise  
 $\hat{\xi}_{\pi} \propto \hat{\xi}_{\varphi}$

but not adiabatic  
in USR

# non-adiabatic transition SR $\rightarrow$ USR Starobinsky (1992) Jackson et al (2023)

mode equation  $u_k'' + \left(k^2 - \frac{z''}{z}\right) u_k = 0$

$$u_k = z R_k = a \delta\phi_k$$



general solution in SR / USR limit

$$\Rightarrow u_k \approx \frac{1}{\sqrt{2k}} \left\{ \alpha_k \left(1 - \frac{i}{k\eta}\right) e^{-ik\eta} + \beta_k \left(1 + \frac{i}{k\eta}\right) e^{ik\eta} \right\}$$

before transition: BD vacuum,  $\alpha_k = 1$ ,  $\beta_k = 0$

after transition: particle production on sub-Hubble scales  
non-adiabatic perts on super-Hubble scales

gradient terms  
at sudden transition

$$\alpha_k \neq 0, \quad \beta_k \propto \left(\frac{k_\tau}{k}\right)^3 \quad \text{for } k \ll k_\tau$$

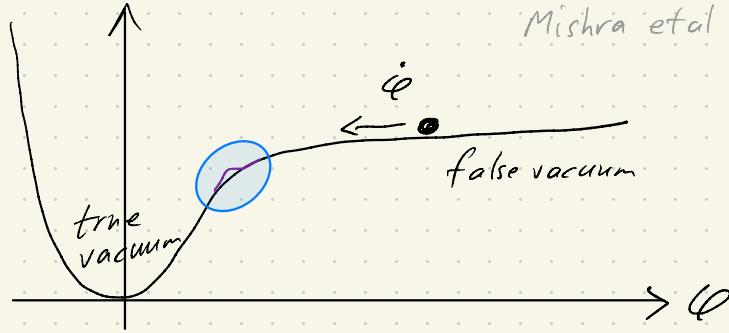
# primordial density perturbations in stochastic VSR

single scalar field,  $\varphi(t)$

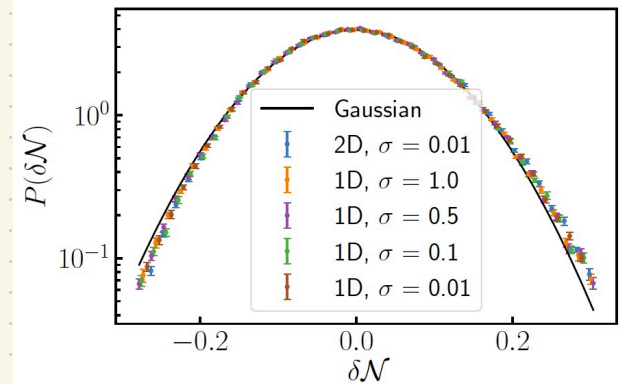
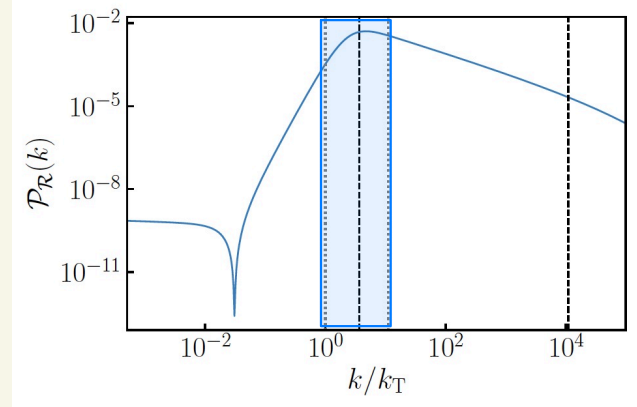
Jackson et al 2410.13683

potential energy,  $V(\varphi)$  + tiny bump

Mishra et al (2024)



typical  
fluctuations  
 $\approx$  Gaussian



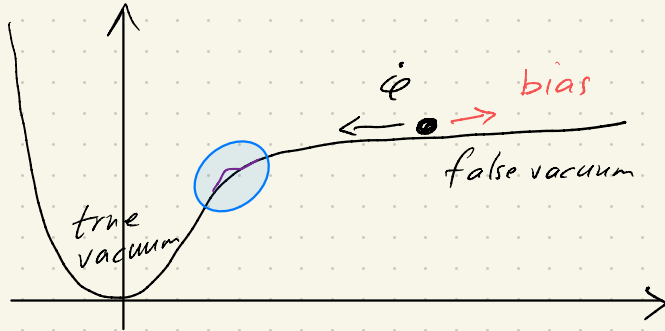
smoothing scale  $k = z_{\text{eff}}$

# importance sampling in stochastic USR

Jackson et al 2410.13683

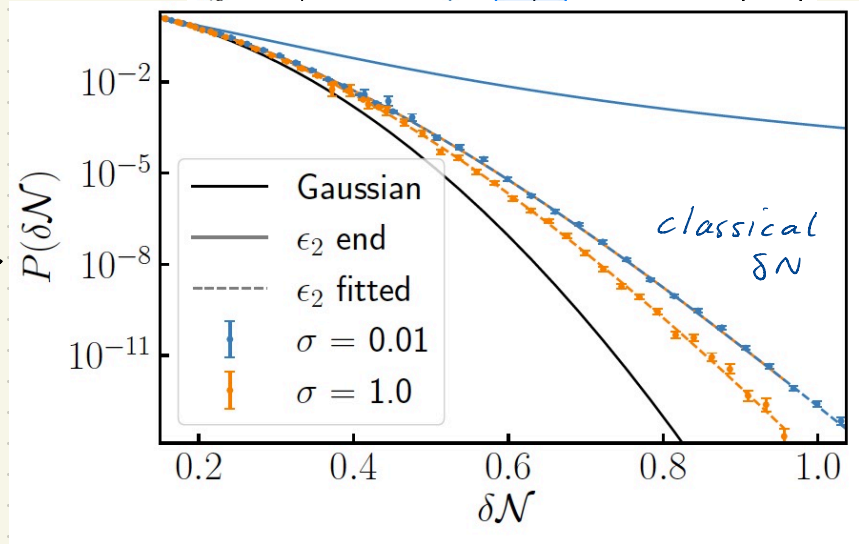
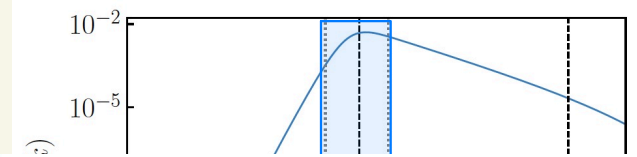
single scalar field,  $\varphi(t)$

potential energy,  $V(\varphi)$  + tiny bump



bias dynamics

→ rare fluctuations



but

- noise calculated only along classical background
- result depends on coarse-graining scale,  $k = 3 \text{ aH}$

# + leading-order gradient terms as coloured noise

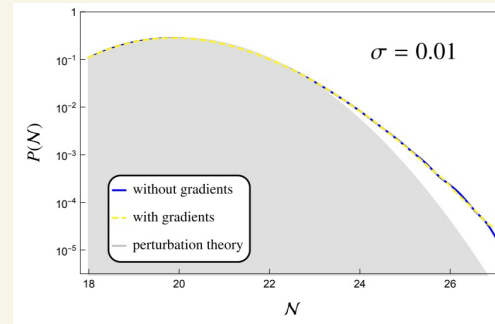
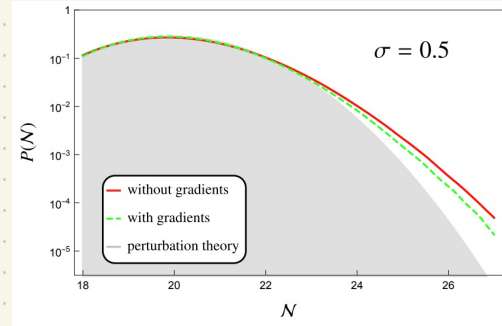
Briaud, Kawaguchi & Vennin (2025)

gradient terms only affect noise in stochastic approach

$$\frac{d\varphi}{dN} = \frac{\pi}{a^3 H} + \hat{\xi}_{\varphi}$$

$$\frac{d\pi}{dN} = -\frac{a^3 V'}{H} + \hat{\xi}_{\pi} + \sum_{i=1}^4 \xi_{\Delta}^{(i)}$$

$$\frac{d\xi_{\Delta}^{(i)}}{dN} = \alpha^{(i)} \xi_{\Delta}^{(i)} + \beta^{(i)} \hat{\xi}_{\varphi} + \gamma^{(i)} \hat{\xi}_{\pi}$$



✓ evolves through sudden transition

✓ eliminates dependence on coarse-graining scale

## conclusions

slow-roll inflation  $\rightarrow$  adiabatic perturbations in CMB & LSS

ultra-slow-roll inflation

$\rightarrow$  non-adiabatic fluctuations on super-Hubble scales

$\rightarrow$  enhanced density perturbations on small scales

$\rightarrow$  primordial black holes + grav. waves

## stochastic $\delta N$

- non-perturbative evolution on super-Hubble scales

$\rightarrow P(\delta N)$ , complementary to classical  $\delta N$

- separate universes in VSR  $\neq$  Hamilton-Jacobi,  $H(\varphi)$

- importance sampling probes non-Gaussian tail

- stochastic noises include leading-order gradients

