Constraints on inflation from the gravitational path integral

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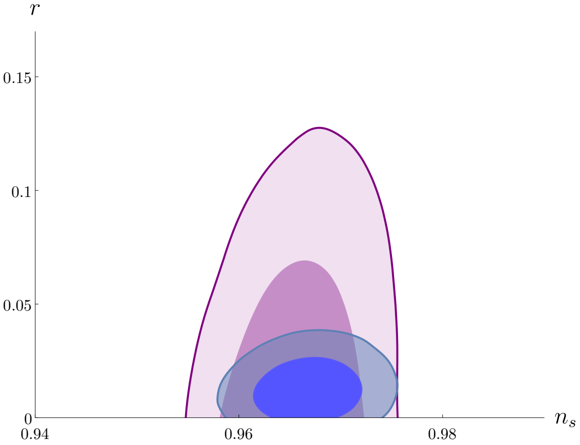
a contribution to Inflation 2025

CMB constraints on precision observables

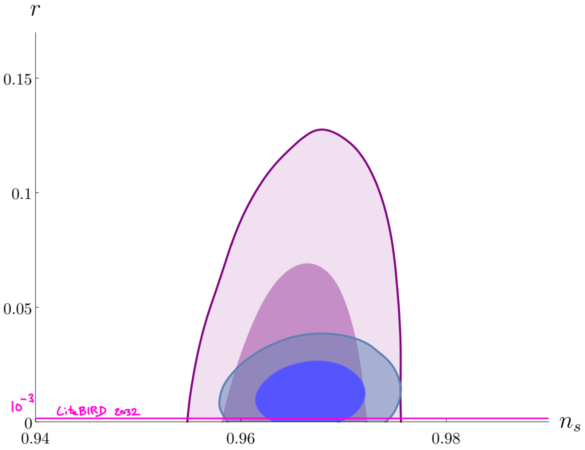
Inflation, like any other theory, requires some parameters as input that need to be measured



[Planck + BICEP/Keck + BAO, 2018]

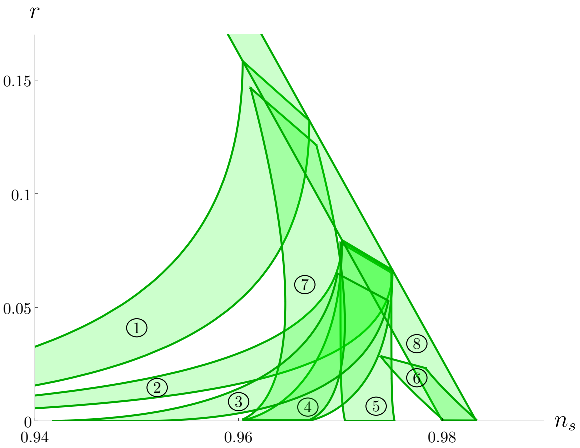
CMB constraints on precision observables

There are exciting near-future experimental prospects



A zoo of models

On the theory side, there are many models. It is far from clear that measuring $(n_s, r, \cdots)$ will tell us what mechanism drove inflation



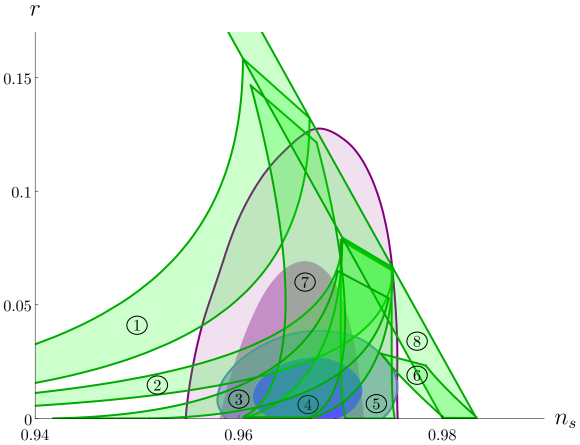
A zoo of models

$$V(\phi) = c_n \cdot \textcircled{n}$$

$$\begin{array}{ll}
① & 1 + \cos(\phi/f) \\
② & 1 - \phi^2/\mu^2 \\
③ & 1 - \phi^4/\mu^4 \\
④ & 1 - \exp(-q\phi) \\
⑤ & 1 - \mu^2/\phi^2 \\
⑥ & 1 + \alpha \log \phi \\
⑦ & \left[1 - \exp\left(-\sqrt{2}\phi/\sqrt{3\alpha}\right)\right]^2 \\
⑧ & \phi^p
\end{array}$$

In each model the parameter is varied in some interval, e.g. $p \in [1/2, 7/2]$, and c_n is fixed so $\Delta T/T \sim 10^{-5}$

A zoo of models



We are missing a guide of the theory landscape

Our motivation

Is there a *theoretical* prior on inflation models?

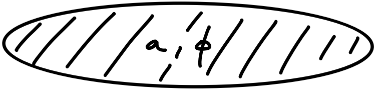
Are some initial conditions for inflation preferred, or perhaps ruled out?

Some approaches:

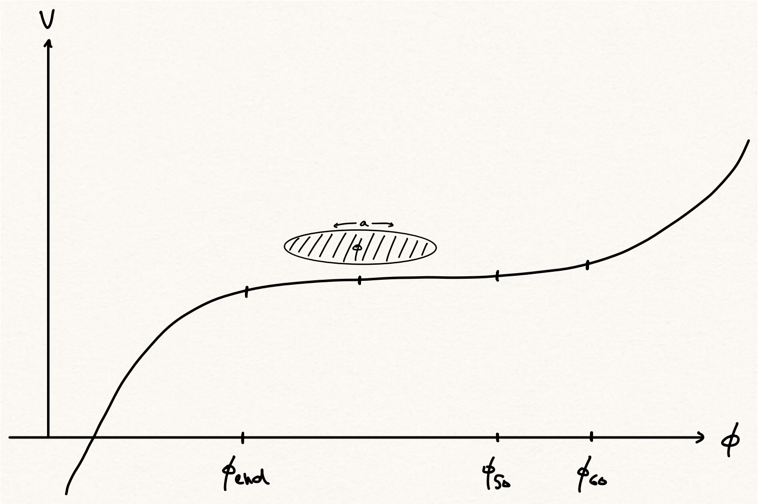
1. Top $\rightarrow$ down: study string inflation
2. Bottom $\rightarrow$ up: not every EFT has a good UV completion
3. **Gravitational path integral**

The GPI of inflation

We will consider a time slice of an inflationary history for a closed universe, which has some size a and is covered by a scalar field at value ϕ



The GPI of inflation



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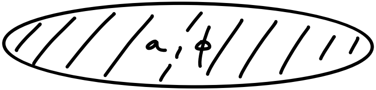
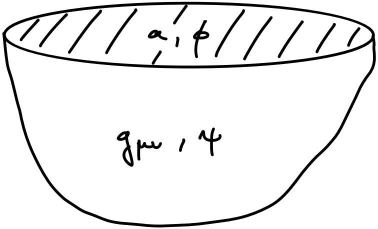


The quantum amplitude for this slice to occur is $\Psi(a, \phi)$, with probability $|\Psi(a, \phi)|^2$. If Ψ is a semiclassical state, the momenta π_a, π_ϕ are also fixed to high accuracy, so then this can be viewed as a probability of initial (or final) conditions for classical evolution

The GPI of inflation

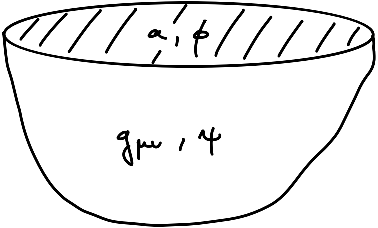
A proposal [Hartle and Hawking '83] for $\Psi(a, \phi)$, in the limit $G_N \to 0$, is by having (a, ϕ) be the boundary values of a regular solution to the equations of motion, on a manifold with only one boundary, with amplitude set by the classical action



$$\Psi_{\text{HH}}(a, \phi) \sim \sum_{M:\partial M = S^3} \int_{(g_{\mu\nu}, \psi)|_{\partial M} = (a, \phi)} \mathcal{D}g_{\mu\nu}\mathcal{D}\psi \; e^{iS[g, \psi; G_N]}$$

$$\sim \sum_J P_J \, e^{iS[g_J, \psi_J; G_N]} \left[1 + \mathcal{O}(G_N)\right] \quad \text{as } G_N \to 0$$

The GPI of inflation



This is certainly not the only possibility, but it has some generality to it, and some mathematical elegance.
"The boundary conditions for the universe are that there are none"

The GPI of inflation

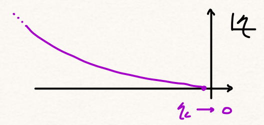
Standard inflationary correlator calculations work in the same way [Maldacena '02, '24]. In flat slicing:

- Spatial metric $\mathrm{d}\Sigma^2 = a^2 e^{2\zeta(\boldsymbol{x})}\mathrm{d}\boldsymbol{x}^2$, we want $\Psi_0[\zeta(\boldsymbol{x})]$ for $|\zeta| \ll 1$
- Write $\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + a(t)^2 e^{2\tilde{\zeta}(t,\boldsymbol{x})}\delta_{ij}\left(N^i\mathrm{d}t + \mathrm{d}x^i\right)\left(N^j\mathrm{d}t + \mathrm{d}x^j\right),\ \phi(t)$
  Then $(\cdots)$

$$S[\tilde{\zeta}] = \int \mathrm{d}^4x\ a(\eta)^2\varepsilon(\eta)\left(\tilde{\zeta}'^2 - (\nabla\tilde{\zeta})^2\right) + \mathcal{O}(\tilde{\zeta}^3)$$

  where $\varepsilon = -\dot{H}/H^2, H = \dot{a}/a$. Conformal time: $\mathrm{d}t = a\,\mathrm{d}\eta$
- Tree-level approximation obtained by action of *complex* classical solution with "outgoing" boundary conditions:

$$|\Psi_0[\zeta]|^2 \sim \exp\left(-\int_{\boldsymbol{k}} \frac{2\varepsilon_*}{H_*^2}k^3\zeta_{-\boldsymbol{k}}\zeta_{\boldsymbol{k}} + \cdots\right), \quad k_{\text{phys}} = \frac{k}{a_*} \sim H_*$$

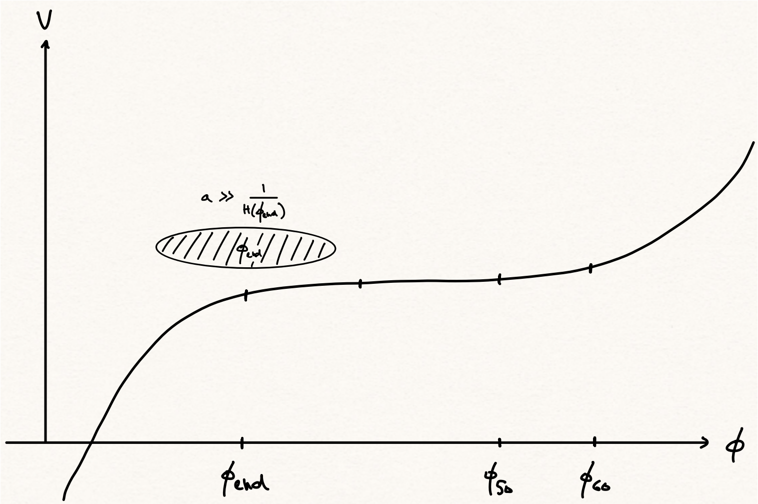


The GPI of inflation

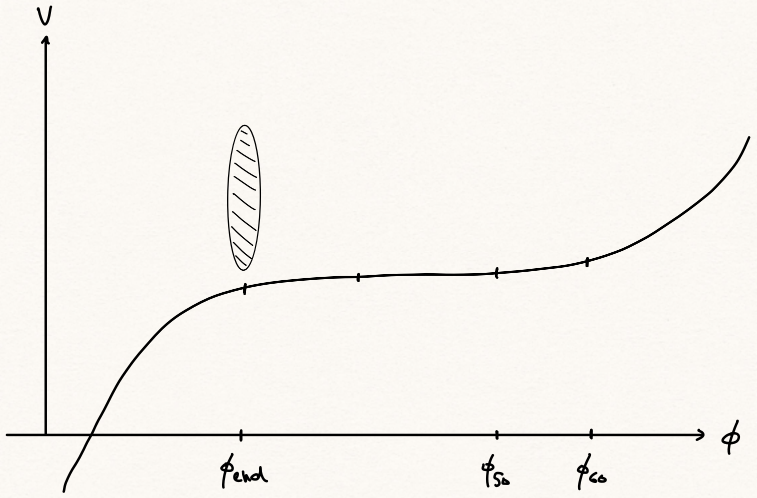
- $\rightsquigarrow$

$$\langle \zeta_{\boldsymbol{k}} \zeta_{\boldsymbol{k}'} \rangle = \frac{\delta\left(\boldsymbol{k}+\boldsymbol{k}'\right)}{k^3} A_s \left(\frac{k}{k_*}\right)^{n_s-1}, \quad n_s = 1 - 2\varepsilon_* - \frac{\dot{\varepsilon}}{\varepsilon H}\bigg|_*$$

Semiclassical limit: saddle points



Semiclassical limit: saddle points



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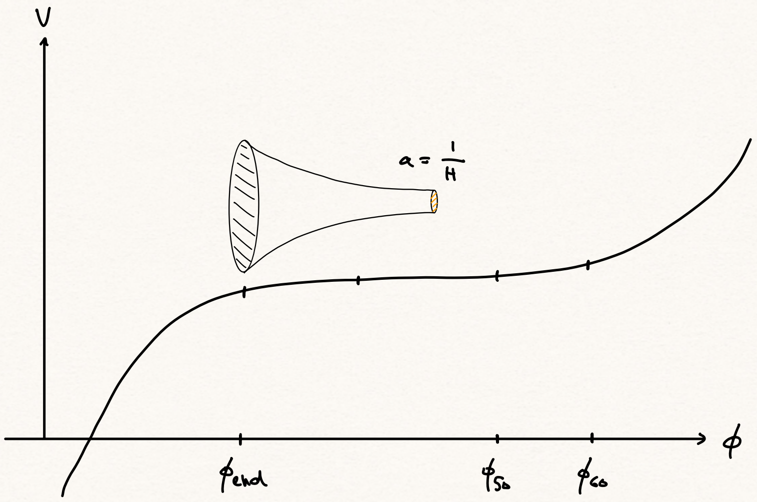
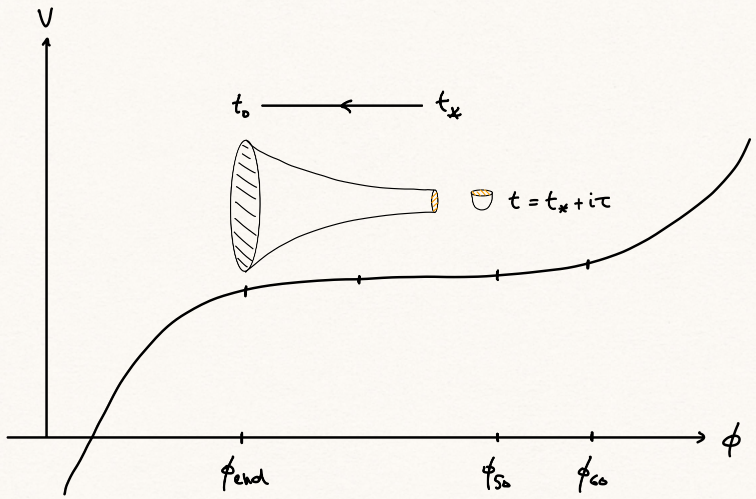
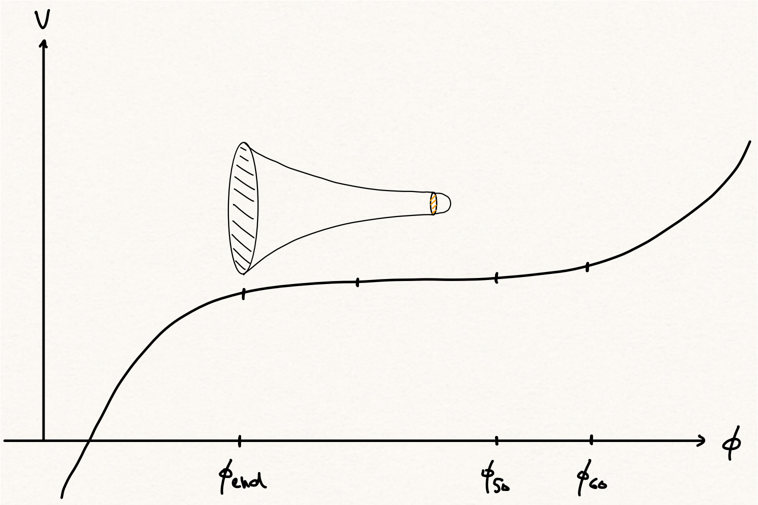


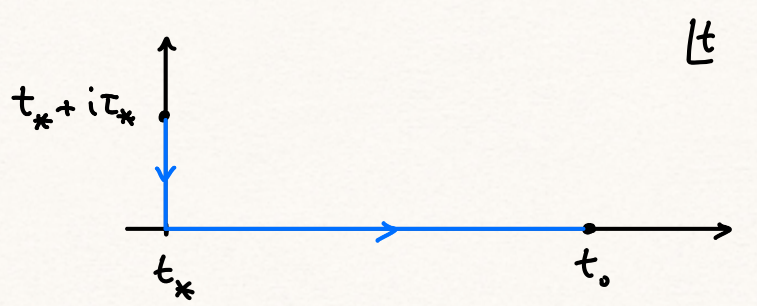
Semiclassical limit: saddle points



Semiclassical limit: saddle points

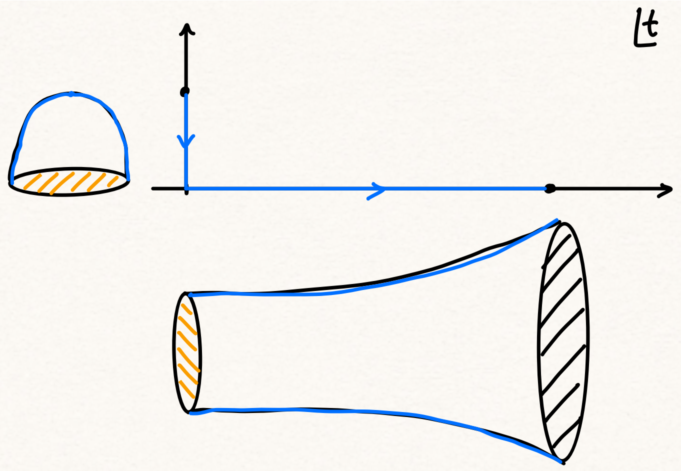


Semiclassical limit: saddle points



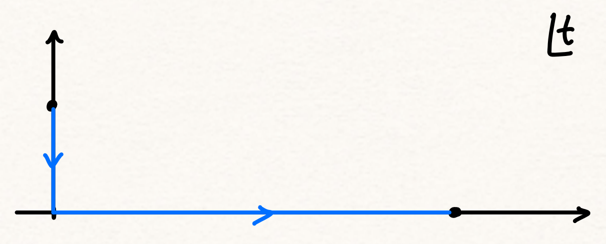
$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \mathrm{d}\Omega_3^2 \,, \quad \phi(t)$$

Semiclassical limit: saddle points



$$a(t) \approx \frac{1}{H_*}\cosh(H_* t)\,,\ \ \phi(t) \approx \phi_*$$

The saddles are complex



$$a(t) = \frac{1}{H_*}\cosh(H_* t)\left[1 + \varepsilon_* \gamma(t) + \mathcal{O}(\varepsilon^2)\right]\ ,\ \phi(t) = \phi_* + \sqrt{2\varepsilon_*}\varphi(t) + \cdots$$

$$\varphi(t) = \frac{1 + i \sinh t}{\cosh^2 t} - \log\left(1 - i \sinh t\right) - \frac{i\pi}{2}$$

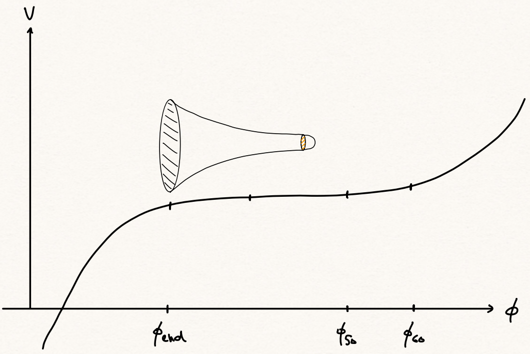
$$= -H_* t + \log 2 + 3e^{-2H_* t} - \frac{16i}{3} e^{-3H_* t} + \cdots \quad \text{as } t \to \infty$$

[Maldacena '24] [OJ '20, '24]

A puzzle

$$|\Psi_{\rm HH}(a,\phi)|^2 \sim \exp\left(-2\,{\rm Im}S\right) = \exp\left(\frac{\pi}{G_N H_*^2}\right)$$

here H_* is the Hubble scale *at the beginning of inflation* ($aH = 1$)



A puzzle

$$\begin{aligned} |\Psi_{\mathrm{HH}}(a,\phi)|^2 &\sim \exp\left(-2\,\mathrm{Im}S\right) \\ &= \exp\left(\frac{\pi}{G_N H_*^2}\right) \end{aligned}$$

This formula favors a small amount of inflation, with a huge pressure:

$$\begin{aligned} \frac{|\Psi_{\mathrm{HH}}(a,\phi)|^2}{|\Psi_{\mathrm{HH}}(a\times e,\phi)|^2} &= \exp\left[\frac{\pi}{G_N}\left(\frac{1}{H_a^2}-\frac{1}{H_{ea}^2}\right)\right] \\ &\approx \exp\left(\frac{2\pi}{G_N}\frac{\varepsilon}{H_a^2}\right) \\ &\sim \exp\left(\frac{c}{A_s}\right) \sim e^{10^9} \end{aligned}$$

A puzzle

$$|\Psi_{\mathrm{HH}}(a,\phi)|^2 \sim \exp\left(-2\,\mathrm{Im}S\right) = \exp\left(\frac{\pi}{G_N H_*^2}\right)$$

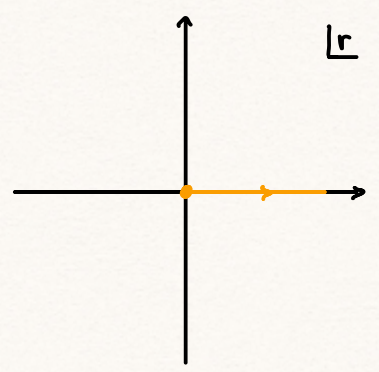
This is a *puzzle* because the Hartle-Hawking proposal gives correct predictions for small inhomogeneous fluctuations (e.g. $\zeta_{\boldsymbol{k}}$) around inflationary background. But it gives the wrong answer for the zero mode $\boldsymbol{k} = \mathbf{0}$! [Maldacena '24]

We will not solve this puzzle. Instead we will assume lots of inflation happened (say, 60 efolds).
The question we're interested in is: which inflation?

$\mathbb{C}$-metrics are weird

Consider the flat metric on $\mathbb{R}^{d+1}$ in spherical coordinates

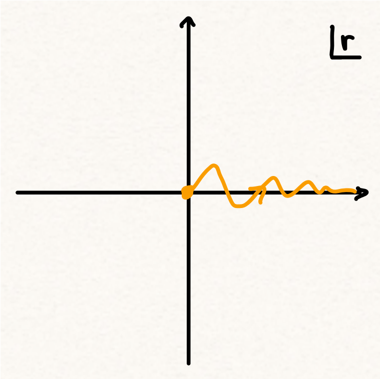
$$ds^2 = dr^2 + r^2 d\Omega_d^2$$



$\mathbb{C}$-metrics are weird

We can deform this into the complex r-plane by letting $r = \gamma(\ell)$, so

$$ds^2 = \gamma'(\ell)^2 d\ell^2 + r(\gamma(\ell))^2 d\Omega_d^2$$



This is still topologically $\mathbb{R}^{d+1}$

$\mathbb{C}$-metrics are weird

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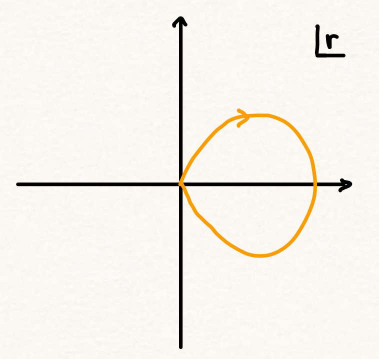
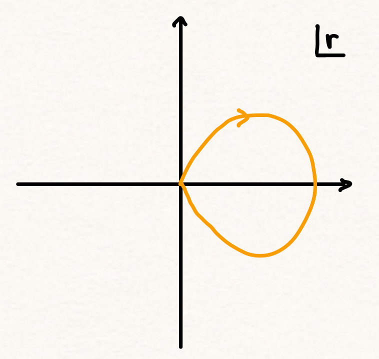


Now this is $\cong S^{d+1}$, but it's complex

$\mathbb{C}$-metrics are weird



This is a flat metric ($R_{\mu\nu\rho\sigma} = 0$) on the sphere. It does not satisfy Gauss-Bonnet for $d = 1$, $\int R = 0 \neq 8\pi$

But some are probably OK

Let's consider a general complex geometry (g, M) and put a massive probe scalar field on it. Let's assume we can compute its correlation functions by performing a path integral over *real* field configurations

$$\langle \phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle = \int_{C_{\mathbb{R}}(M)} \mathcal{D}\phi \; \phi(x_1)\cdots\phi(x_n)\, e^{-S[\phi;g]}$$

where

$$S = \frac{1}{2}\int_M \mathrm{d}^D x \sqrt{\det g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right)$$

This manifestly converges if, for all $\phi \in C_{\mathbb{R}}(M)$ and $x \in M$,

$$\mathrm{Re}\left[\sqrt{\det g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right)\right] > 0$$

But some are probably OK

We can also consider more general probe, free p-form matter A_p with field strength $F = \mathrm{d}A_p$ and ask

$$\mathrm{Re}\,[F \wedge \star F] > 0 \qquad \text{on } M, \text{ for all } A_p,\ p \in \{-1, 0, \cdots, D\}$$

[Kontsevich & Segal '21] [Witten '21]

The idea is that QFT is well-defined on such backgrounds

KSW for a practical person

Consider a metric g on M, with components $g_{\mu\nu}(x) \in \mathbb{C}$ in your favorite real coordinate basis. KSW is equivalent to passing the following algorithm [Benetti Genolini, OJ, Murthy; to appear]

KSW for a practical person

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1. $\det g$ is nowhere negative real on M. *Define* $\mathrm{Re}\sqrt{\det g(x)} > 0$
2. $A \equiv \mathrm{Re}\left(\sqrt{\det g}\, g^{-1}\right)$ is positive definite everywhere on M
3. $$\sum_{i=1}^{D} |\arg[\lambda_i(g(x)A(x))]| < \pi$$

   for all $x \in M$

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   for all $x \in M$

This can be used to (numerically) rule out $\mathbb{C}$-metrics

Application to the GPI of inflation

The Hartle-Hawking saddle point has an allowable representation

$$\Updownarrow$$

$$\frac{V'(\phi_*)}{V(\phi_*)} \int_{\phi_*}^{\phi_{\text{end}}} |\mathrm{d}\phi| \frac{V'(\phi_*)}{V'(\phi)} < 1 + O(\varepsilon_*)$$

[OJ, '24]

Application to the GPI of inflation

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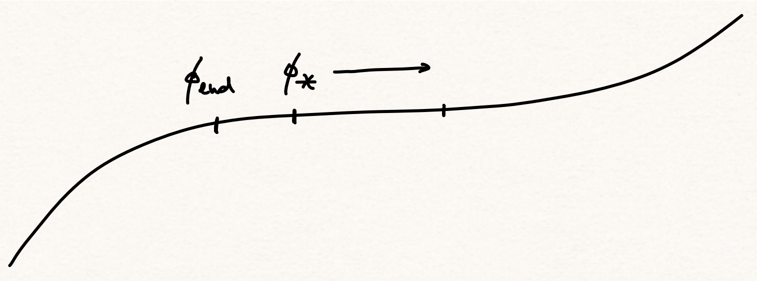

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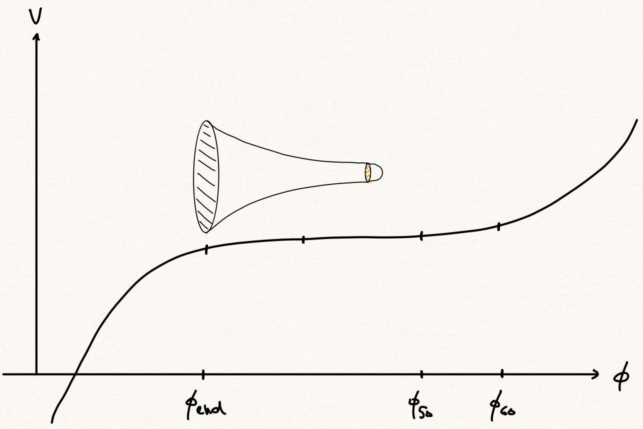

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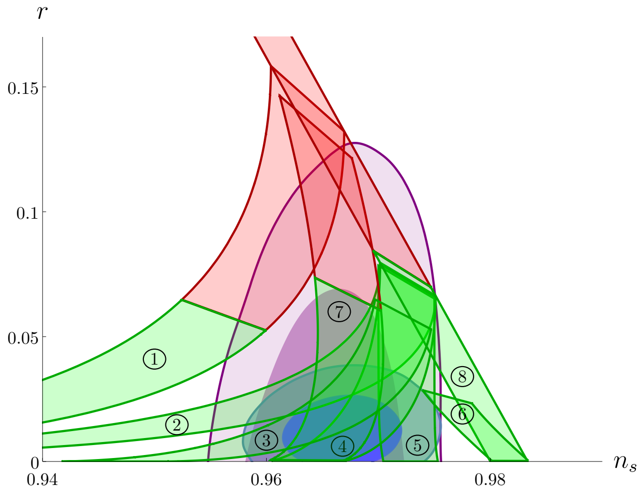


Thank you for your attention!

Recap of KSW idea

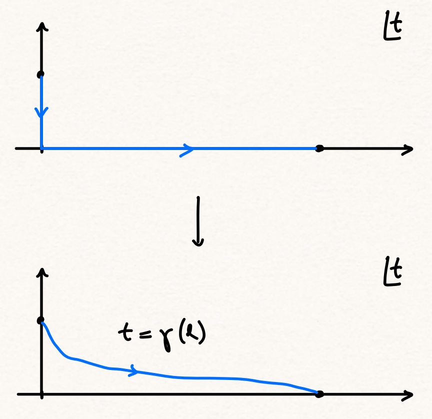
We are trying to evaluate a GPI in the semiclassical limit $G_N \to 0$. We assume that however it is defined, as $G_N \to 0$ its main contributions will come from the vicinity of saddles (solutions to the equations of motion)

Which saddles are relevant, though, depends on what the theory is (far) away from the saddles. For quantum gravity in our universe we don't know this theory

The idea though is that we may already be able to tell which saddles *are not* relevant, by trying to integrate over other fields while fixing the gravitational part to its saddle point "value" (configuration)

This will likely become moot when we understand quantum gravity better

Allowable representation of Hartle-Hawking saddle



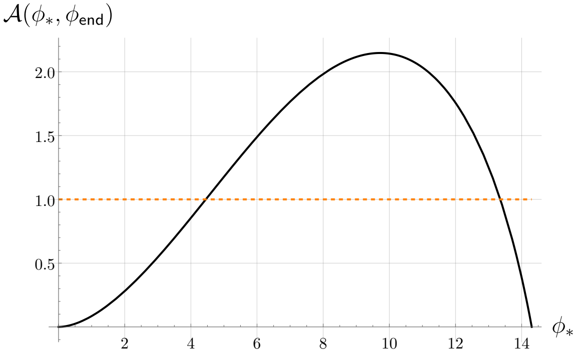
$$
\begin{aligned}
ds^2 &= g_{\mu\nu}(x)dx^\mu dx^\nu \\
&= -dt^2 + a(t)^2 d\Omega_3^2 \\
&= -\gamma'(\ell)^2 d\ell^2 + a(\gamma(\ell))^2 d\Omega_3^2
\end{aligned}
$$

Does there *exist* a curve γ, connecting the south pole to the boundary, with

$$\sum_{i=1}^{4} |\arg \lambda_i(\ell)| < \pi \quad \text{for all } \ell?$$

KSW & large curvature puzzle of the HH state

$$f = 5: \quad N_e \lesssim 0.9 \text{ or } N_e \gtrsim 42$$



$$V(\phi) = c_f \left[1 + \cos(\phi/f)\right]$$

$$\mathcal{A} = \frac{V'(\phi_*)}{V(\phi_*)} \int_{\phi_*}^{\phi_{\mathsf{end}}} |\mathrm{d}\phi| \frac{V'(\phi_*)}{V'(\phi)}$$