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A Tale of Two Potentials

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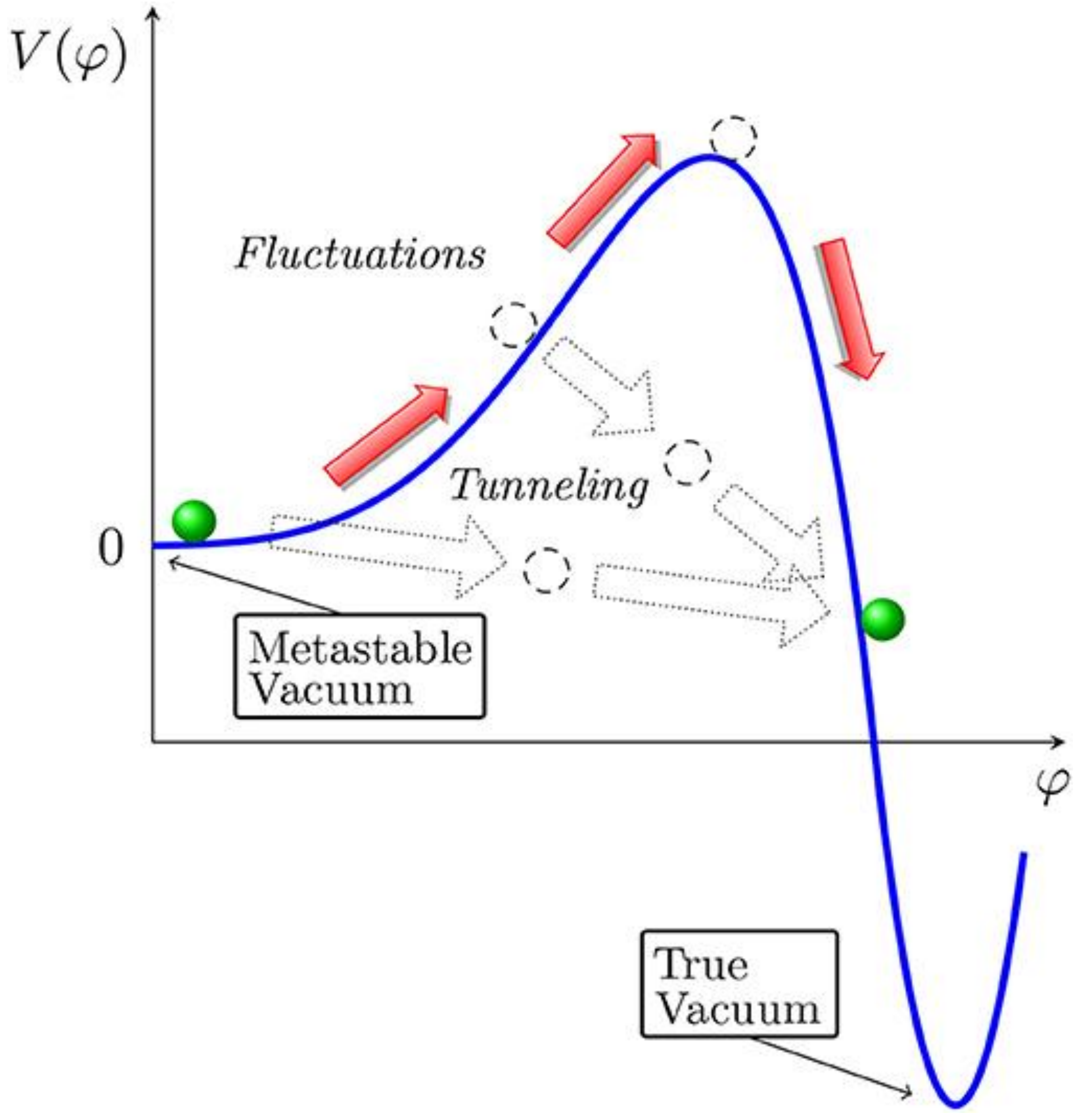
Inflation 2025, Paris
3/12/2025

Effective potential in cosmology

The scalar field effective potential is ubiquitous in cosmology:

- Inflationary dynamics
- Dark energy
- Phase transitions
- Vacuum decay

Whenever we use the (semi)classical equations of motion to describe the field dynamics,
we replace the quantum field theory potential (or action) with some kind of an effective potential (or action)



Stochastic theory

Stochastic Langevin equation for long-wavelength dynamics of light scalar fields in de Sitter (Starobinsky 1986; Starobinsky&Yokoyama 1994):

$$\frac{d\bar{\phi}}{dt} + \frac{\mathcal{V}'(\bar{\phi})}{3H} = \xi, \qquad \langle \xi(t)\xi(t') \rangle = \frac{H^3}{4\pi^2}\delta(t-t')$$

Benefit: Can be solved non-perturbatively

- Exact one-point probability distribution
- Numerically exact long-distance correlation functions using spectral expansion (Markkanen, AR, Stopyra & Tenkanen 2018)

What is $\mathcal{V}(\bar{\phi})$?

- ✘ Not the bare QFT potential because it is UV divergent
- ✘ Not the renormalised QFT potential because it is renormalisation scheme dependent

⇒ Some kind of an effective potential

But the effective potential is not unique in de Sitter!

Standard effective potential

Wick rotation to imaginary time $t \to i\tau$
Partition function in presence of a (constant) external field J:

$$Z(J) = \int \mathcal{D}\phi \, e^{-S[\phi] + J\int d^4x \sqrt{g}\phi}$$

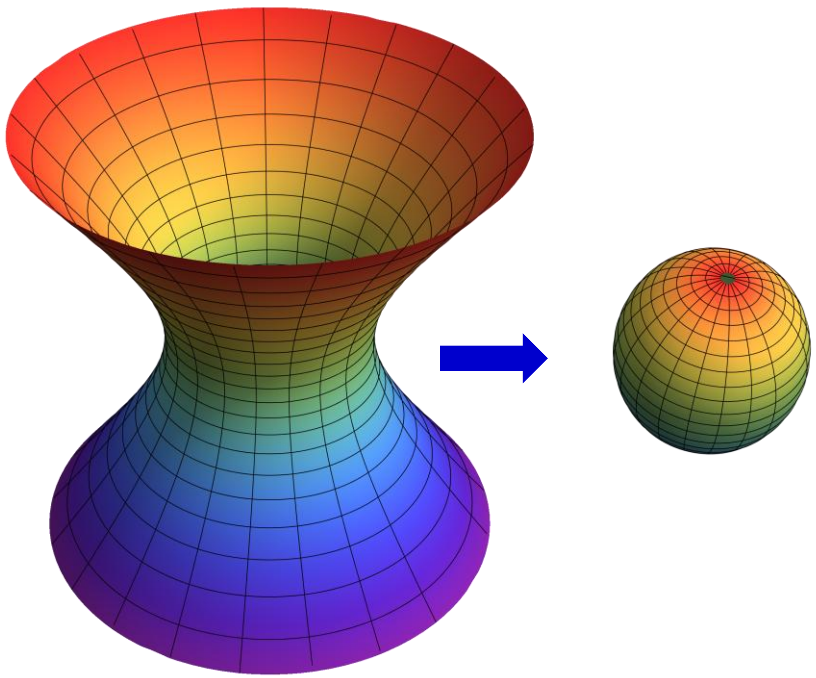
Define $W(J) = -\frac{1}{V}\ln Z(J)$, where V is the spacetime volume

The standard effective potential $\mathcal{V}_S(\bar{\phi})$ is given by the Legendre transform
(DeWitt 1964; Jona-Lasinio 1964; Jackiw 1974):

$$\mathcal{V}_S(\bar{\phi}) = -W(J) + J\bar{\phi}, \qquad \bar{\phi} = -W'(J)$$

- Or, equivalently, by the sum of one-particle irreducible vacuum Feynman diagrams (Goldstone, Salam & Weinberg 1962; Jackiw 1974)
- Minimum determines the vacuum expectation value of the field: $\mathcal{V}_S'(\bar{\phi}) = 0$ for $\bar{\phi} = \langle\phi\rangle$
- Coefficients of $\bar{\phi}^n$ give quantum corrected vertices

Scalar field in de Sitter



De Sitter spacetime

$$ds^2 = -dt^2 + \frac{\cosh^2 Ht}{H^2} d\Omega_3^2$$

- Good approximation for inflationary spacetime
- H = Hubble rate

Wick rotation → four-sphere with radius $1/H$:

$$ds^2 = d\tau^2 + \frac{\cosh^2 H\tau}{H^2} d\Omega_3^2$$

Action on four-sphere

$$S_E = \int d^4x\sqrt{g}\left[\frac{1}{2}Z_\phi \partial_\mu\phi\partial^\mu\phi + \frac{1}{2}(m^2+\delta m^2)\phi^2 + \frac{1}{2}(\xi+\delta\xi)R\phi^2 + \frac{1}{4}(\lambda+\delta\lambda)\phi^4\right]$$

- Dimensional regularisation and $\overline{\mathrm{MS}}$ renormalisation: The same counterterms as in Minkowski
- Constant curvature $R = 12H^2 \Rightarrow$ Absorb ξ to curvature dependent mass $M^2 = m^2 + 12\xi H^2$

One-loop standard effective potential in de Sitter

One-loop approximation:

$$\mathcal{V}_S(\bar{\phi}) = \frac{1}{2}(M^2 + \delta M^2)\bar{\phi}^2 + \frac{1}{4}(\lambda + \delta\lambda)\bar{\phi}^4 + \frac{3H^4}{16\pi^2}\ln\det\left(\frac{\Delta_{S^4} + M^2 + 3\lambda\bar{\phi}^2}{H^2}\right)$$

$$= \frac{1}{2}(M^2 + \delta M^2)\bar{\phi}^2 + \frac{1}{4}(\lambda + \delta\lambda)\bar{\phi}^4 + \frac{3H^4}{16\pi^2}\sum_{n=0}^{\infty} d_n \ln\frac{\omega_n^2}{H^2}$$

where the eigenvalues ω_n^2 and degeneracies d_n are

$$\omega_n^2 = (n^2 + (d-1)n)H^2 + V''(\bar{\phi}), \qquad d_n = \frac{(2n + d - 1)\Gamma(n + d - 1)}{\Gamma(n+1)\Gamma(d)}, \qquad n \geq 0$$

- Note: $\omega_0^2 = V''(\bar{\phi})$
- The sum can be evaluated analytically in $d = 4 - \epsilon$ dimensions
- Counterterms cancel the UV divergences

One-loop standard effective potential in de Sitter

The full one-loop result is

$$\mathcal{V}_S(\bar{\phi}) = \frac{1}{2}M^2\bar{\phi}^2 + \frac{1}{4}\lambda\bar{\phi}^4$$
$$-\frac{H^4}{64\pi^2}\left\{\left(\nu^4 - \frac{\nu^2}{2}\right)\left(\ln\frac{\mu^2}{H^2} + 1\right) + 6\nu^2 - (4\nu^2 - \nu)\ln\frac{\Gamma\left(\frac{3}{2}+\nu\right)}{\Gamma\left(\frac{3}{2}-\nu\right)}\right.$$
$$+(12\nu^2 - 1)\left[\psi^{(-2)}\left(\frac{3}{2}+\nu\right) + \psi^{(-2)}\left(\frac{3}{2}-\nu\right)\right]$$
$$-24\nu\left[\psi^{(-3)}\left(\frac{3}{2}+\nu\right) - \psi^{(-3)}\left(\frac{3}{2}-\nu\right)\right]$$
$$\left.+24\left[\psi^{(-4)}\left(\frac{3}{2}+\nu\right) + \psi^{(-4)}\left(\frac{3}{2}-\nu\right)\right]\right\},$$

where $\nu = \sqrt{\frac{9}{4} - \frac{V''(\bar{\phi})}{H^2}}$

Light field expansion of the standard effective potential

Expand

$$\mathcal{V}_S(\bar{\phi}) = \frac{1}{2}M_S^2\bar{\phi}^2 + \frac{1}{4}\lambda_S\bar{\phi}^4 + \frac{1}{6}\kappa_S\bar{\phi}^6 + \mathcal{O}(\bar{\phi}^8)$$

for light fields $M^2 \ll H^2$ (Garcia-Consuegra&AR 2025):

$$M_S^2 = M^2 + \frac{3\lambda H^2}{16\pi^2}\left[\frac{3H^2}{M^2} + \ln\frac{\mu^2}{H^2} + 2\gamma - \frac{23}{6} - \frac{M^2}{2H^2}\left(\ln\frac{\mu^2}{H^2} + 2\gamma + \frac{2}{27}\right) + \mathcal{O}\left(\frac{M^4}{H^4}\right)\right],$$

$$\lambda_S = \lambda - \frac{9\lambda^2}{16\pi^2}\left[\frac{6H^4}{M^4} + \ln\frac{\mu^2}{H^2} + 2\gamma + \frac{2}{27} - \frac{M^2}{2H^2}\left(\frac{46}{243} + \frac{16\zeta(3)}{9}\right) + \mathcal{O}\left(\frac{M^4}{H^4}\right)\right],$$

$$\kappa_S = \frac{27\lambda^3}{32\pi^2H^2}\left[\frac{12H^6}{M^6} + \frac{46}{243} + \frac{8\zeta(3)}{9} - \frac{M^2}{2H^2}\left(\frac{40\zeta(3)}{27} - \frac{176}{729}\right) + \mathcal{O}\left(\frac{M^4}{H^4}\right)\right]$$

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- Infrared contributions ⇒ Expansion parameter $\lambda H^4/M^4$
- Resum? But is this actually the quantity we want?

Constraint effective potential

Proposed by O'Raifeartaigh et al. in 1986:

$$\mathcal{V}_C(\bar{\phi}) = -\frac{1}{V}\ln\int\mathcal{D}\phi\,\delta\left(\bar{\phi} - \frac{1}{V}\int d^4x\sqrt{g}\phi(x)\right)e^{-S[\phi]}$$

Equivalent to the standard effective potential in infinite volume, but not in de Sitter

One-point probability distribution

$$p(\bar{\phi}) \propto \exp\left(-\frac{8\pi^2\mathcal{V}_C(\bar{\phi})}{3H^4}\right) \Rightarrow \langle\phi\rangle = \int d\bar{\phi}\bar{\phi}p(\bar{\phi})$$

One-loop calculation (Garcia-Consuegra&AR 2025):

Delta functional removes the $n = 0$ mode from the sum

$$\mathcal{V}_C(\bar{\phi}) = \frac{1}{2}(M^2 + \delta M^2)\phi^2 + \frac{1}{4}(\lambda + \delta\lambda)\phi^4 + \frac{3H^4}{16\pi^2}\left[\ln\det\left(\frac{\Delta_{S^4} + M^2 + 3\lambda\bar{\phi}^2}{H^2}\right) - \ln\frac{M^2 + 3\lambda\bar{\phi}^2}{H^2}\right]$$

$$= \mathcal{V}_S(\bar{\phi}) - \frac{3H^4}{16\pi^2}\ln\left(1 + \frac{3\lambda\bar{\phi}^2}{M^2}\right)$$

Light field expansion of the constraint effective potential

Expand

$$\mathcal{V}_C(\bar{\phi}) = \frac{1}{2}M_C^2\bar{\phi}^2 + \frac{1}{4}\lambda_C\bar{\phi}^4 + \frac{1}{6}\kappa_C\bar{\phi}^6 + \mathcal{O}(\bar{\phi}^8)$$

for light fields $M^2 \ll H^2$ (Garcia-Consuegra&AR 2025):

$$M_C^2 = M^2 + \frac{3\lambda H^2}{16\pi^2}\left[\ln\frac{\mu^2}{H^2} + 2\gamma - \frac{23}{6} - \frac{M^2}{2H^2}\left(\ln\frac{\mu^2}{H^2} + 2\gamma + \frac{2}{27}\right) + \mathcal{O}\left(\frac{M^4}{H^4}\right)\right],$$

$$\lambda_C = \lambda - \frac{9\lambda^2}{16\pi^2}\left[\ln\frac{\mu^2}{H^2} + 2\gamma + \frac{2}{27} - \frac{M^2}{2H^2}\left(\frac{46}{243} + \frac{16\zeta(3)}{9}\right) + \mathcal{O}\left(\frac{M^4}{H^4}\right)\right],$$

$$\kappa_C = \frac{27\lambda^3}{32\pi^2 H^2}\left[\frac{46}{243} + \frac{8\zeta(3)}{9} - \frac{M^2}{2H^2}\left(\frac{40\zeta(3)}{27} - \frac{176}{729}\right) + \mathcal{O}\left(\frac{M^4}{H^4}\right)\right]$$

No infrared contributions / negative powers of M^2:

- Expansion parameter $\lambda \ll 1 \Rightarrow$ Can be computed perturbatively
- Valid for $M^2/H^2 \to 0$

Link to stochastic theory

Quantum field theory

- One-point probability distribution

$$p_{\mathrm{QFT}}(\bar{\phi}) \propto \exp\left(-\frac{8\pi^2 \mathcal{V}_C(\bar{\phi})}{3H^4}\right)$$

Stochastic theory

- One-point probability distribution

$$p_{\mathrm{stoch}}(\bar{\phi}) \propto \exp\left(-\frac{8\pi^2 \mathcal{V}(\bar{\phi})}{3H^4}\right)$$

Does this mean we should choose $\mathcal{V}(\bar{\phi}) = \mathcal{V}_C(\bar{\phi})$?

If we do, then

- Two-point correlators match at one loop for light fields (Garcia-Consuegra&AR 2025):

$$\langle \mathcal{T}\phi(x)\phi(y)\rangle_{\mathrm{QFT}} = \langle \bar{\phi}(x)\bar{\phi}(y)\rangle_{\mathrm{stoch}} + \mathcal{O}(\lambda^2)$$

- The lifetime of a metastable vacuum calculated using the Hawking-Moss instanton in QFT matches with the saddle-point approximation in the stochastic theory (Camargo-Molina, Carrillo Gonzalez & AR 2023):

$$\Gamma_{\mathrm{QFT}} = \Gamma_{\mathrm{stoch}}$$

Summary

In de Sitter, the definition of the scalar field effective potential is not unique

Constraint effective potential (O'Raifeartaigh, Wipf & Yoneyama 1986):

$$\mathcal{V}_C(\bar{\phi}) = -\frac{1}{V}\ln\int\mathcal{D}\phi\,\delta\left(\bar{\phi} - \frac{1}{V}\int d^4x\sqrt{g}\phi(x)\right)e^{-S[\phi]}$$

- Appears to be the correct potential to use with Starobinsky-Yokoyama stochastic theory:
  - ✓ One-point probability distribution
  - ✓ One-loop two-point correlator for light fields
  - ✓ Hawking-Moss vacuum decay rate

- No infrared problem ⇒ Can be computed perturbatively

- May also be calculable using lattice Monte Carlo simulations

- For more details, see arXiv:2511.23076