# Primordial Black Hole formation from a massless scalar field

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SMASH


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# INTRODUCTION

- **Primordial Black Holes** (PBHs) are formed in the very early Universe spanning a wide range of masses in order of magnitudes: PBHs are **interesting candidates for Dark Matter, Gravitational waves (LVK signals), seeds of SMBHs**.

- Cosmological perturbations with an amplitude larger than threshold $\delta_c$ collapse into PBHs after re-entering the cosmological horizon.

- The aim of this work is to study the **PBH formation** from the collapse of adiabatic perturbations of a massless scalar field (massive field in the future).

- This scenario was investigated by *T. Harada & B. Carr:* Growth of primordial black holes in a universe containing a massless scalar field, PRD (2005)

# MATHEMATICAL FORMALISM

- **Space time metric:** 3+1 conformal decomposition with the metric line element:

$$ds^2 = -\alpha^2 dt^2 + a^2(t) e^{2\zeta} \tilde{\gamma}_{ij} \left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)$$

- **Stress-energy tensor for a massless scalar field:**

$$T_{\mu\nu} = -\frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi + \partial_\mu \phi \partial_\nu \phi$$

- **Conjugate momentum:** $$\Pi = \frac{1}{\alpha}\left(\partial_t - \beta^i \partial_i\right)\phi$$

- **Klein-Gordon equation:** $$\begin{cases} \left(\partial_t - \beta^i \partial_i\right)\Pi = \alpha \triangle \phi + \alpha K \Pi + D_i \phi D^i \phi \\ \left(\partial_t - \beta^i \partial_i\right)\phi = \alpha \Pi \end{cases}$$

# SCALAR FIELD vs PERFECT FLUID

- **Cosmic time slicing:** we assume spherical symmetry

$$ds^2 = -A^2(t,r)dt^2 + B^2(t,r)dr^2 + R^2(t,r)d\Omega^2$$

- **Misner-Sharp differential operators:** $\begin{cases} D_t = \frac{1}{A}\partial_t \\ D_r = \frac{1}{B}\partial_r \end{cases} \Rightarrow \begin{cases} \eta = D_t\phi \\ \psi = D_r\phi \end{cases}$

- **Stress-energy tensor:** $T_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}\left(\psi^2 - \eta^2\right) + \partial_\mu\phi\partial_\nu\phi$

In the comoving gauge no flux ($\psi = 0$) $\Rightarrow$ **The scalar field behaves concisely as a perfect fluid, with an equation of state $p = \rho$, ONLY during the nearly linear regime.**

# GRADIENT EXPANSION

- **Gradient expansion:** expand dynamical variables in spatial gradient of the curvature perturbation on super horizon scales where

$$\varepsilon(t) \propto \frac{k}{H(t)a(t)} \ll 1$$

is measuring the order of the gradient expansion.

- **Adiabatic cosmological perturbations:**

$$\phi(t,r) = \phi_b(t)\left[1 + \epsilon^2(t)\tilde{\phi}(r) + O\left(\varepsilon^3\right)\right]$$

$$\Pi(r,t) = \Pi_b(t)\left[1 + \epsilon^2(t)\tilde{\pi}(r) + O\left(\varepsilon^3\right)\right]$$

# CURVATURE PROFILE

- At **super-horizon scales** the Universe is described by the **asymptotic form of the metric**:

$$ds^2 = -dt^2 + a^2(t)e^{2\zeta(r)}\left[dr^2 + r^2 d\Omega^2\right] = -dt^2 + a^2(t)\left[\frac{d\tilde{r}^2}{1 - K(\tilde{r})\tilde{r}^2} + \tilde{r}^2 d\Omega^2\right]$$

- **Perturbation amplitude:** defined as the **peak of the compaction function**

$$C(r) = -r\zeta'(r)\left[2 + r\zeta'(r)\right] \qquad \Rightarrow \qquad \delta := C(r_m) \quad \text{where} \quad C'(r_m) = 0$$

- **The threshold** $\delta_c$ depends on the **shape parameter** $\alpha$

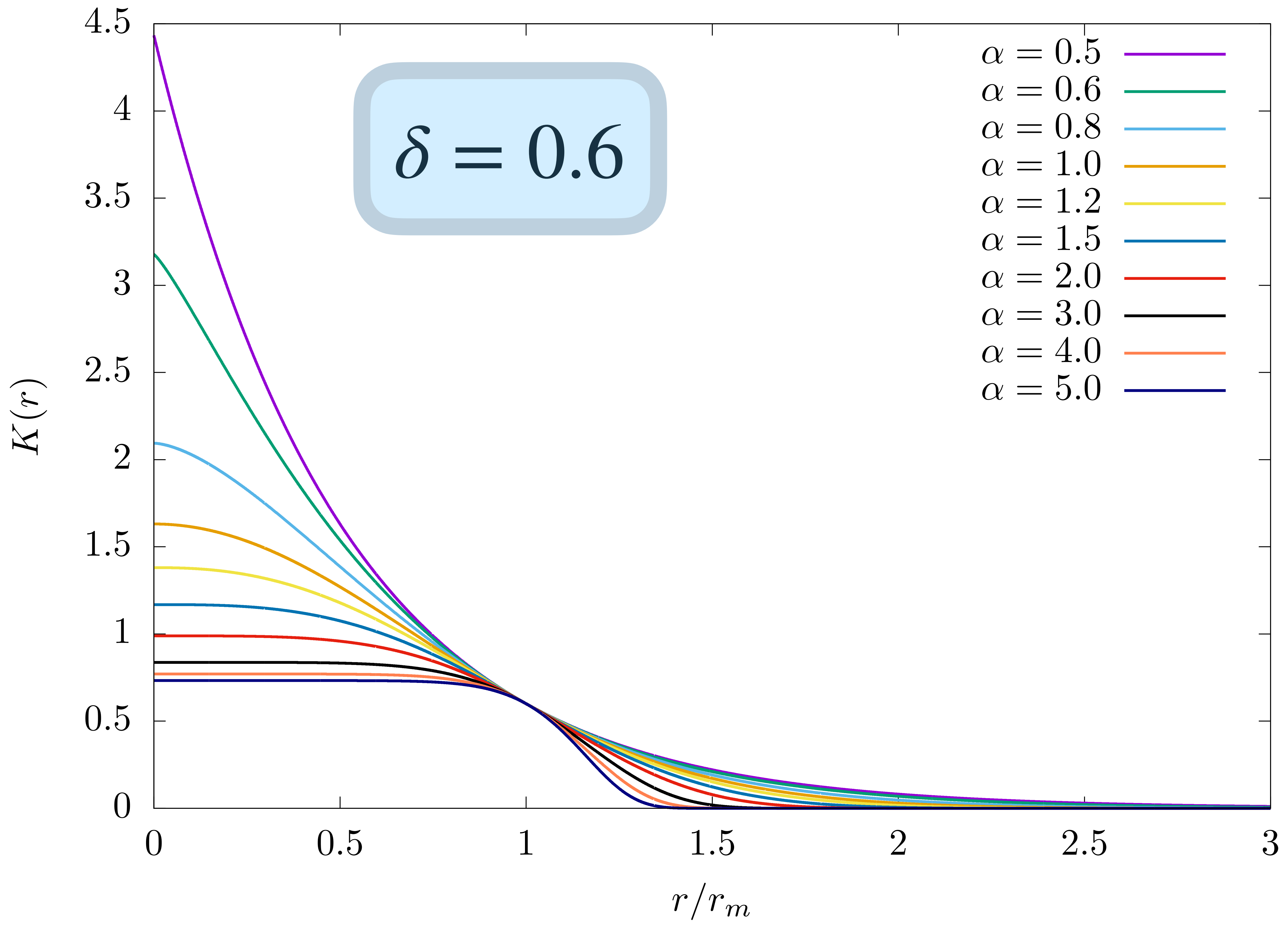
$$\tilde{r} = re^{\zeta(r)}, \quad \alpha = -\frac{C''\left(\tilde{r}_m\right)\tilde{r}_m^2}{4C\left(\tilde{r}_m\right)} \qquad \Rightarrow \qquad \alpha = -\frac{C''\left(r_m\right)r_m^2}{4C\left(r_m\right)\left[1 - \frac{3}{2}C\left(r_m\right)\right]}$$

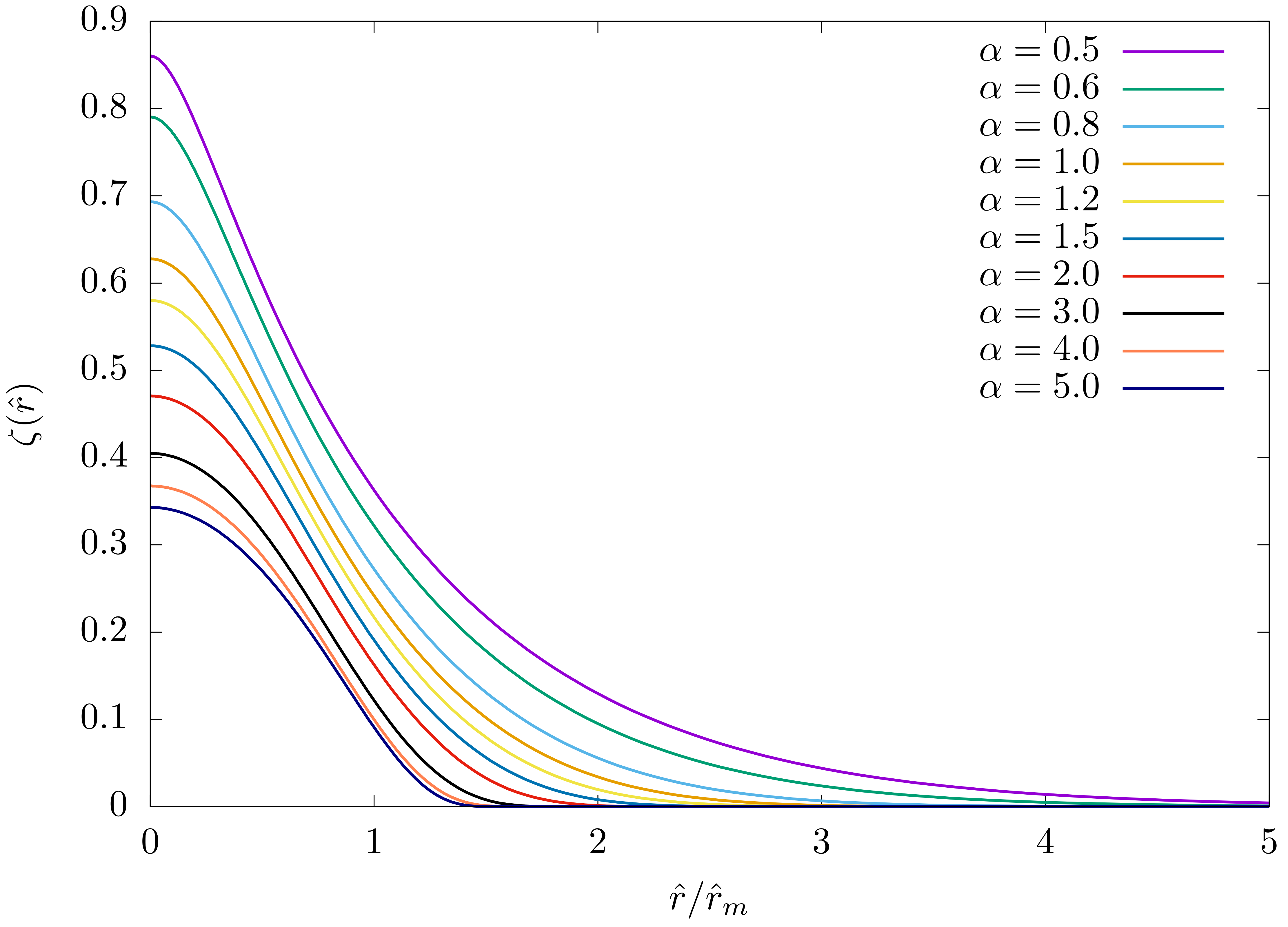
**I. Musco (2019), Escriva et al. (2020), Musco et al. (2021)**

# INITIAL CONDITIONS: CURVATURE PROFILE

$$K(\tilde{r}) = \mathcal{A} \exp\left[-\frac{1}{\alpha}\left(\frac{\tilde{r}}{\tilde{r}_m}\right)^{2\alpha}\right]$$

$$K(\tilde{r}) = -r\zeta'(r)\left[2 + r\zeta'(r)\right]$$

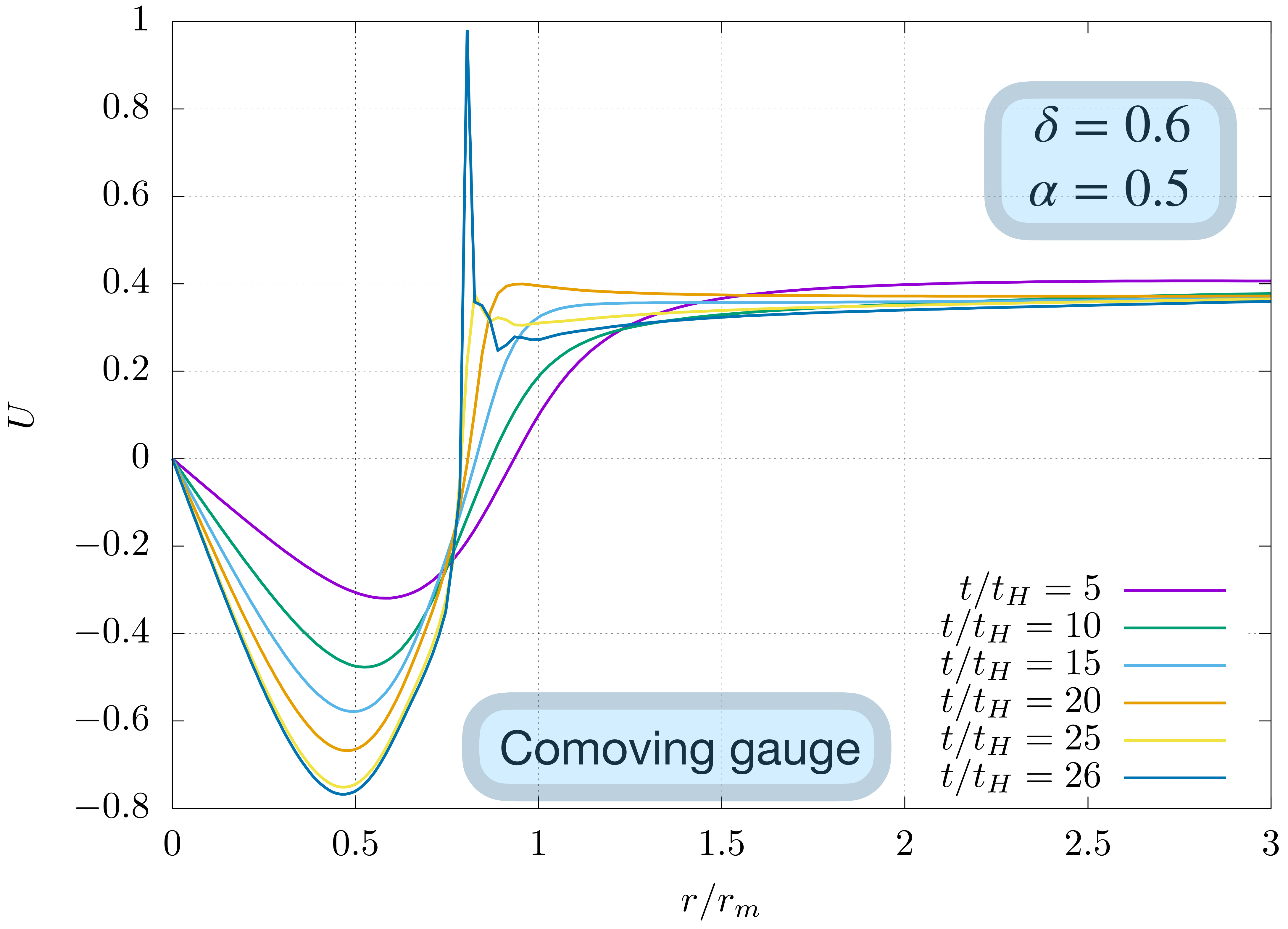


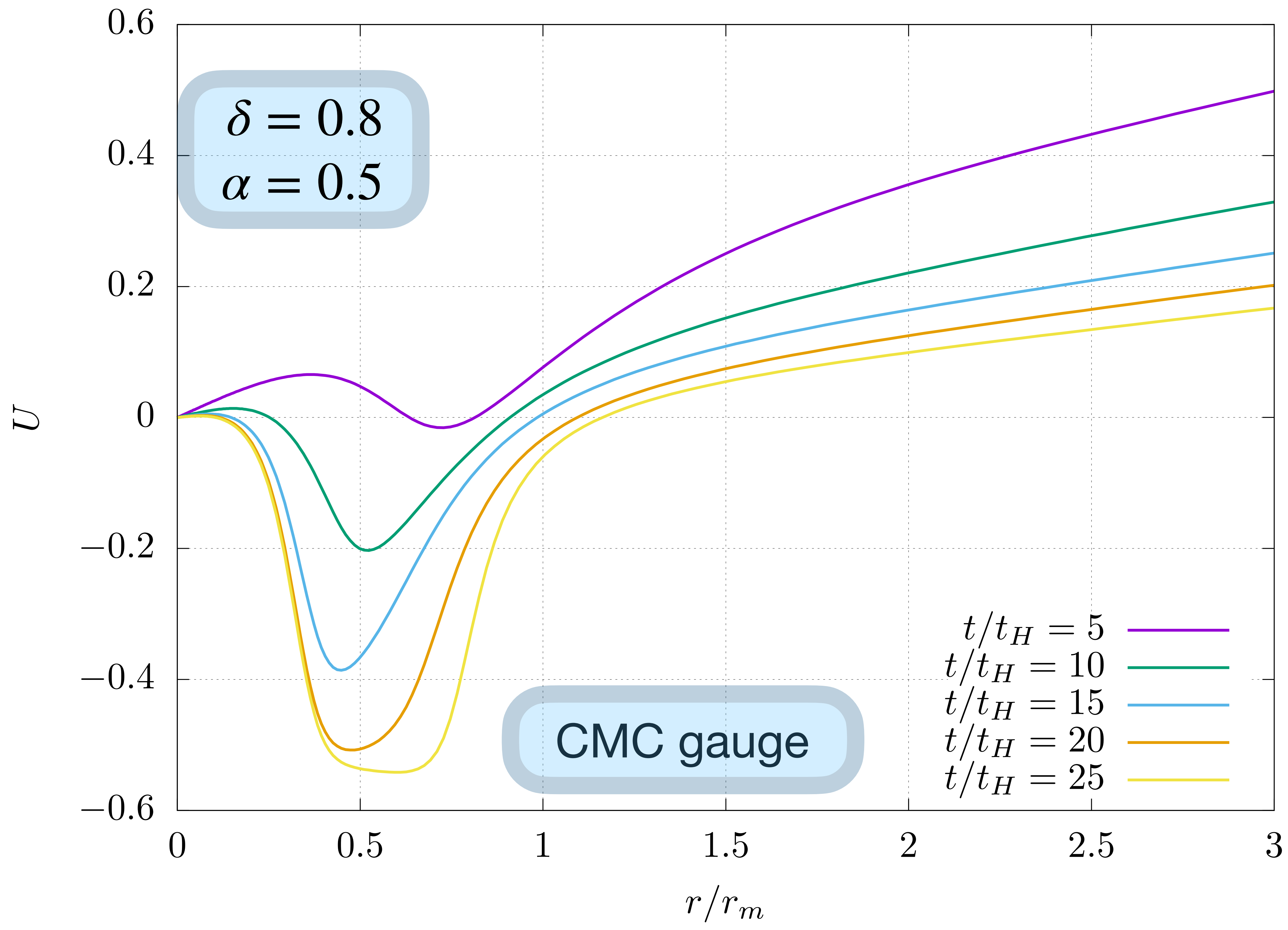



# NUMERICAL SIMULATIONS

- **Spherically symmetric relativistic hydrodynamical code:** BSSN formalism using a non-uniform (exponential growth) grid extended up to 10 perturbation length-scales.

- **Comoving against CMC gauge:** the comoving gauge is failing because of a coordinate singularity.

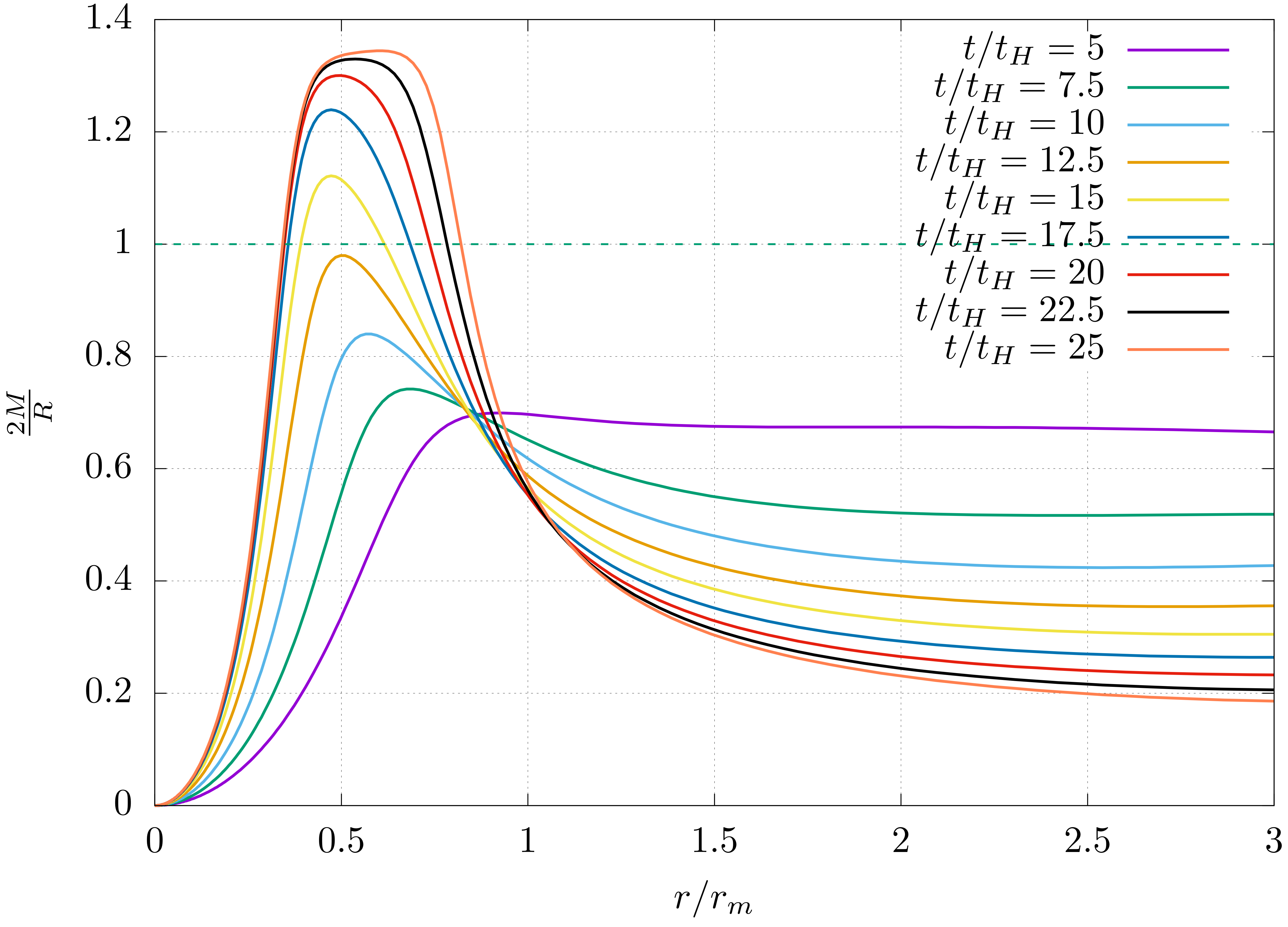


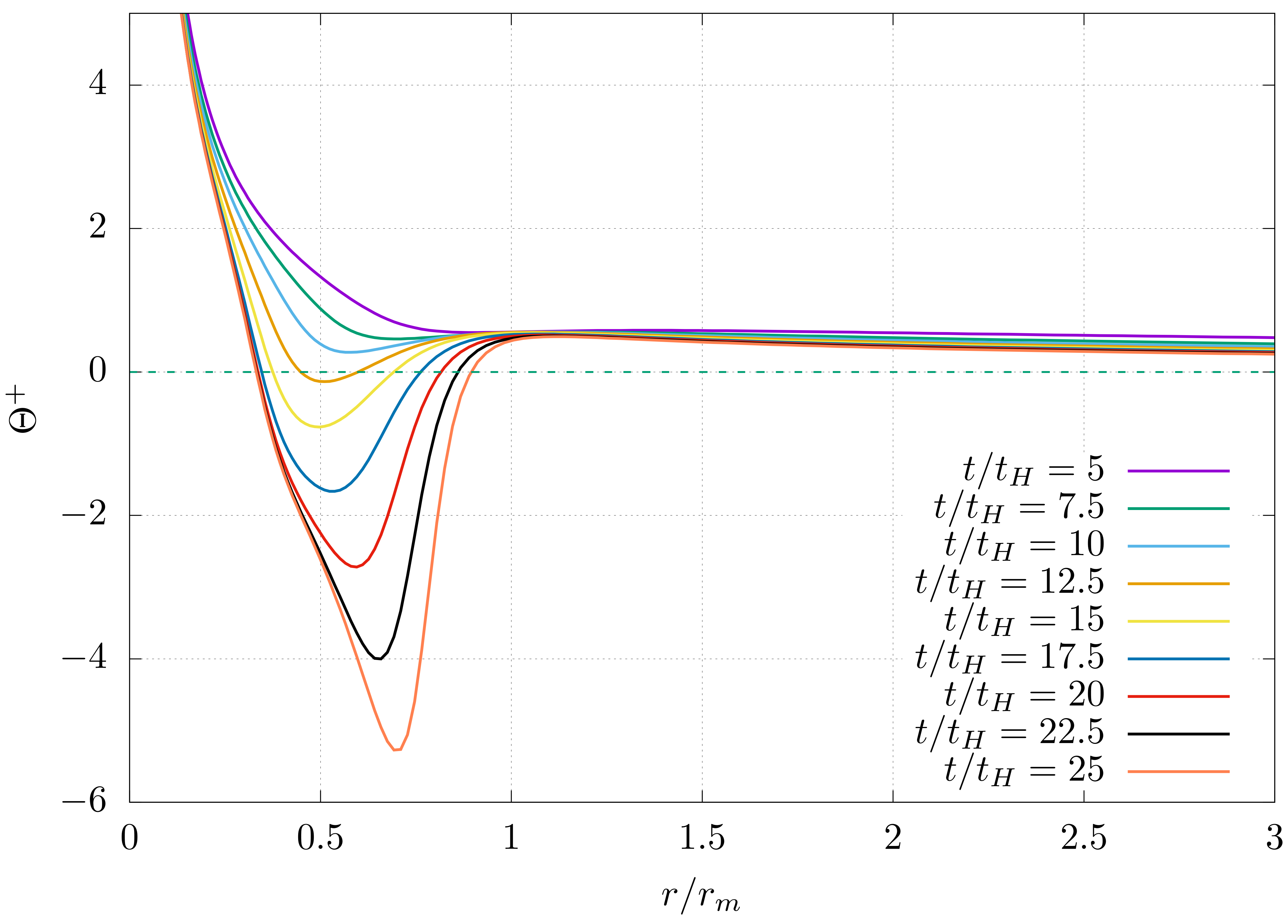



# APPARENT HORIZON FORMATION

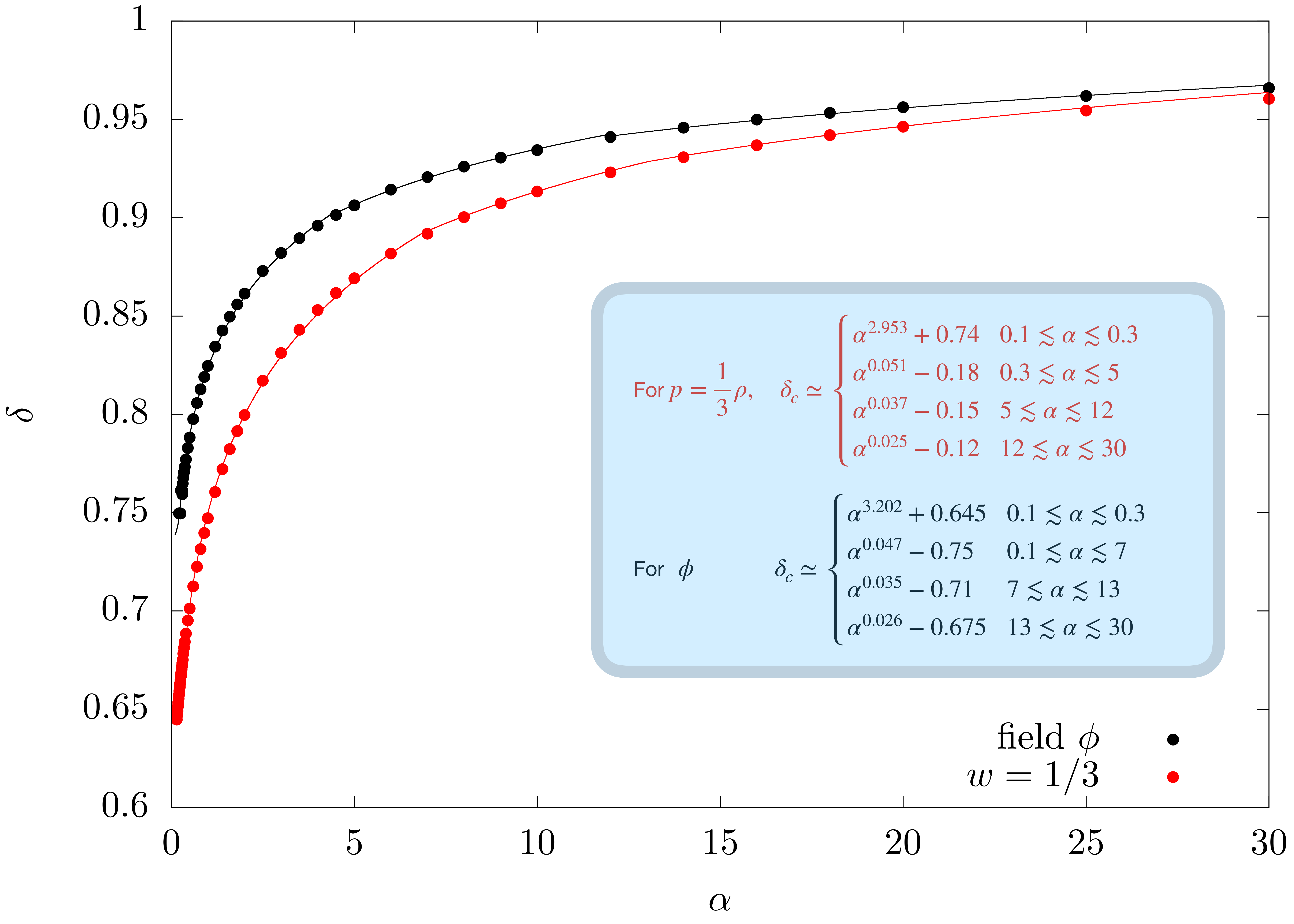
$$R = 2M$$

$$\theta^+ = 0$$

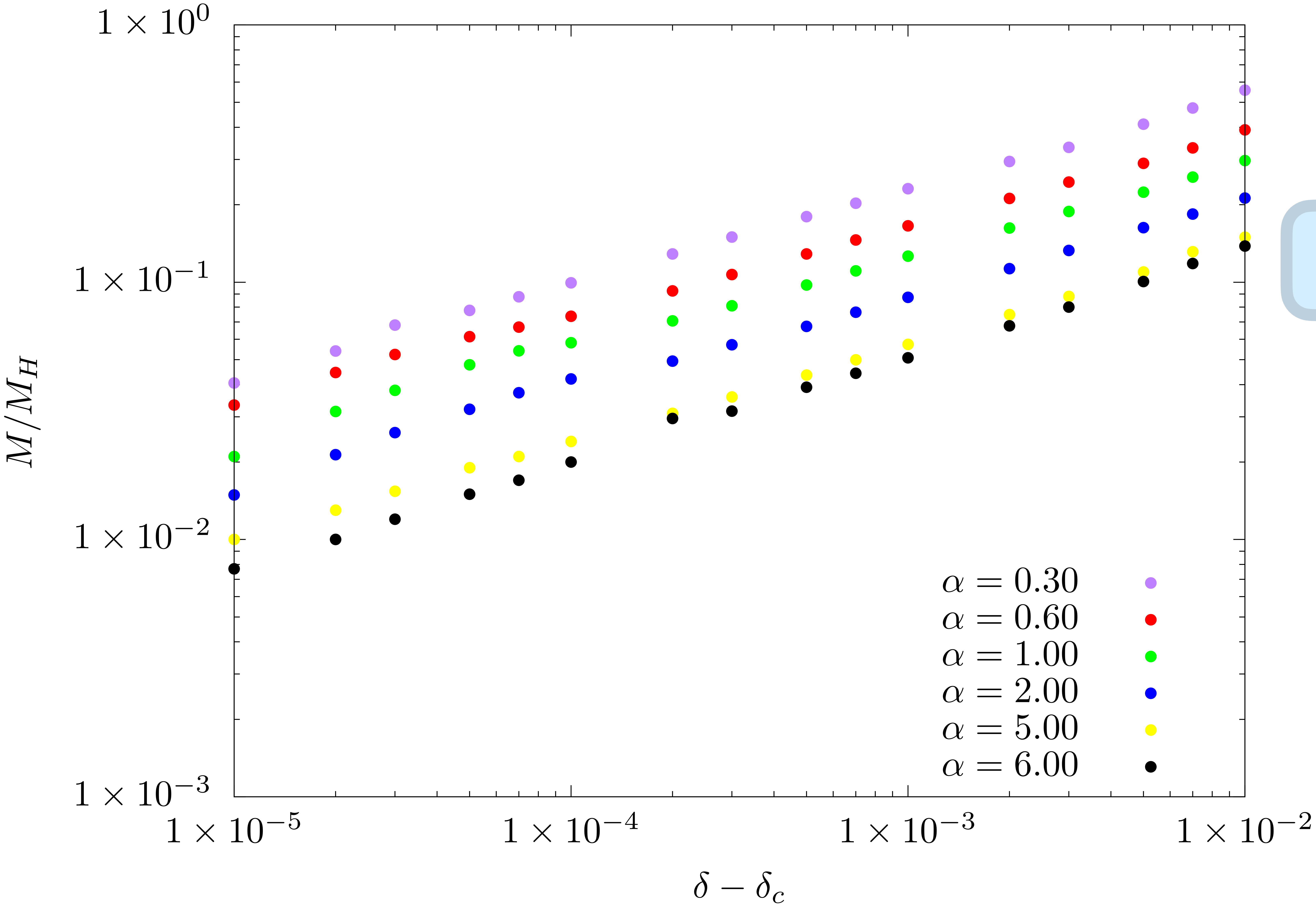


# THRESHOLD FOR PBHs



# CRITICAL COLLAPSE (SCALING LAW)




$$M_{BH} = \mathscr{K}\left(\delta-\delta_c\right)^{0.37} M_H$$

$M_H$ - Horizon Mass

$\mathscr{K}, \delta_c$ - shape dependent

# PBH ABUNDANCE

*C.Germani, IM* - **PRL (2019)**

- **PDF of δ follows a Gaussian distribution:**

$$P(\delta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\delta^2}{2\sigma^2}\right)$$

$$\sigma^2 = \langle \delta^2 \rangle = \int_0^\infty \frac{dk}{k} \mathcal{P}_\delta(k, r_m) = \left(\frac{2}{3}\right)^4 \int_0^\infty \frac{dk}{k} (kr_m)^4 T^2(k, r_m) W^2(k, r_m) \mathcal{P}_\zeta(k)$$

$$\beta_f \simeq \sqrt{\frac{2}{\pi}} \mathcal{K} \left(\frac{k_*}{a_m H_m}\right)^3 \sigma^\gamma \nu_c^{1-\gamma} \gamma^{\gamma+1/2} e^{-\frac{\nu_c^2}{2}} \qquad \nu_c \equiv \frac{\delta_c}{\sigma}$$

- **If** $M_{PBH} \sim 10^{16} g$ are Dark Matter $\Rightarrow \beta_f \simeq 10^{-8} \sqrt{\frac{M_{PBH}}{M_\odot}} \simeq 10^{-16}$
- **Narrow peak:** $\frac{k_*}{\sigma} \gg 1 \;\Rightarrow\; \nu_c \simeq 0.22 \sqrt{\frac{k_*}{\sigma \mathcal{P}_0}} \;\Rightarrow\; \mathcal{P}_0 \sim 7 \times 10^{-4} \frac{k_*}{\sigma} \gg 10^{-3}$
- **Broad peak:** $\frac{k_*}{\sigma} \ll 1 \;\Rightarrow\; \nu_c \simeq 0.46 (\mathcal{P}_0)^{-1/2} \;\Rightarrow\; \mathcal{P}_0 \sim 3 \times 10^{-3}$
- **Non linear effects:** $\delta = \delta_G \left[1 - \frac{3}{8}\delta_G\right] \;\Rightarrow\; 1.5 \lesssim \frac{\mathcal{P}_{0_{NL}}}{\mathcal{P}_{0_L}} = \frac{16\left(1 - \sqrt{1 - \frac{3}{2}\delta_c}\right)^2}{9\delta_c^2} \lesssim 4$

***S.Young, IM, C.Byrnes* JCAP (2019)**

***De Luca, Franciolini, Kehagias, Peloso, Riotto and Unal (2019)***

# PBH threshold prescription

*IM, De Luca, Franciolini, Riotto* - PRD (2021)

Curvature power spectrum $\mathcal{P}_\zeta$

$\Downarrow$

Characteristic overdensity scale $k_*\hat{r}_m$

$\Downarrow$

Characteristic shape parameter $\alpha$

$\Downarrow$

Threshold $\delta_c$

1. **The power spectrum of the curvature perturbation**: take the primordial power spectrum $\mathcal{P}_\zeta$ of the Gaussian curvature perturbation and compute, on superhorizon scales, its convolution with the transfer function $T(k,\eta)$

$$P_\zeta(k,\eta) = \frac{2\pi^2}{k^3}\mathcal{P}_\zeta(k)T^2(k,\eta).$$

2. **The comoving length scale** $\hat{r}_m$ of the perturbation is related to the characteristic scale $k_*$ of the power spectrum $P_\zeta$. Compute the value of $k_*\hat{r}_m$ by solving the following integral equation

$$\int dk k^2 \left[ (k^2\hat{r}_m^2 - 1)\frac{\sin(k\hat{r}_m)}{k\hat{r}_m} + \cos(k\hat{r}_m)\right] P_\zeta(k,\eta) = 0\,.$$

3. **The shape parameter:** compute the corresponding shape parameter $\alpha$ of the collapsing perturbation, including the correction from the non linear effects, by solving the following equation

$$F(\alpha)\left[1+F(\alpha)\right]\alpha = -\frac{1}{2}\left[1+\hat{r}_m\frac{\int dk k^4 \cos(k\hat{r}_m)P_\zeta(k,\eta)}{\int dk k^3 \sin(k\hat{r}_m)P_\zeta(k,\eta)}\right]$$

$$F(\alpha) = \sqrt{1-\frac{2}{5}e^{-1/\alpha}\frac{\alpha^{1-5/2\alpha}}{\Gamma\left(\frac{5}{2\alpha}\right)-\Gamma\left(\frac{5}{2\alpha},\frac{1}{\alpha}\right)}}\,.$$

4. **The threshold $\delta_c$:** compute the threshold as function of $\alpha$, fitting the numerical simulations, at *superhorizon scales*, making a linear extrapolation at horizon crossing ($aHr_m = 1$).

$$\delta_c \simeq \begin{cases} \alpha^{0.047} - 0.50 & 0.1 \lesssim \alpha \lesssim 7 \\ \alpha^{0.035} - 0.475 & 7 \lesssim \alpha \lesssim 13 \\ \alpha^{0.026} - 0.45 & 13 \lesssim \alpha \lesssim 30 \end{cases}$$

# SUMMARY AND FUTURE

To appear very soon…

1. We have investigated the formation of PBHs from a massless scalar field, computing the **threshold**, ad the **mass distribution**, building a new numerical code (3+1 ADM scheme).

2. In the future we would like to extend this analysis for **massive scalar fields** to study different models of dark matters.

3. **These results**, together with PBHs formed in the radiation dominated epoch of the Universe, **will be used, using LSST (Rubin) data** with machine learning techniques (SBI), **to understand to which extent Dark Matter is described by PBHs**.

***Thanks for your attention!***

# 3+1 conformal decomposition

$$\tilde{\triangle}\psi = \frac{\tilde{R}^k_k}{8}\psi - 2\pi\psi^5 a^2 E - \frac{\psi^5 a^2}{8}\left(\tilde{A}_{ij}\tilde{A}^{ij} - \frac{2}{3}K^2\right)$$

$$\tilde{D}^j\left(\psi^6\tilde{A}_{ij}\right) - \frac{2}{3}\psi^6\tilde{D}_i K = 8\pi\psi^6 J_i$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right)\psi = -\frac{\dot{a}}{2a}\psi + \frac{\psi}{6}\left(\overline{\mathcal{D}}_i\beta^i - \alpha K\right)$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right)K = \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\right) - \mathcal{D}^i\mathcal{D}_i\alpha + 4\pi\alpha\,(E+S)$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right)\tilde{A}_{ij} = \frac{1}{a^2\psi^4}\left[\alpha\left(R_{ij} - \frac{1}{3}\gamma_{ij}R_{(sc)}\right) - \left(\mathcal{D}_i\mathcal{D}_j\alpha - \frac{\gamma_{ij}}{3}\mathcal{D}_k\mathcal{D}^k\alpha\right)\right] + \alpha\left(K\tilde{A}_{ij} - 2A_{ik}A^k_j\right) +$$
$$-\frac{2}{3}\tilde{A}_{ij}\overline{\mathcal{D}}_k\beta^k - \frac{8\pi\alpha}{a^2\psi^4}\left(S_{ij} - \frac{\gamma_{ij}}{3}S^k_k\right)$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right)\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} - \frac{2}{3}\tilde{\gamma}_{ij}\overline{\mathcal{D}}_k\beta^k$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right)\Pi = \alpha\triangle\phi + \alpha K\Pi + D_i\phi D^i\phi$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right)\phi = \alpha\Pi$$

# Gradient expansion equations

$$
\begin{cases}
\zeta(t,r) = \zeta_b(r) + \xi(t,r) + O\left(\varepsilon^3\right) \\
\tilde{\gamma}_{ij(t,r)} = \eta_{ij} + h_{ij}(t,r) + O\left(\varepsilon^3\right) \\
K(t,r) = K_b(t)\big(1+\kappa(t,r)\big) + O\left(\varepsilon^3\right) \\
\alpha(t,r) = 1 + \chi(t,r) + O\left(\varepsilon^3\right) \\
\beta^i(t,r) = O\left(\varepsilon\right) \\
\phi(t,r) = \Phi(t) + \lambda(t,r) + O\left(\varepsilon^3\right) \\
\Pi(t,r) = \Pi_b(t) + \Omega(t,r) + O\left(\varepsilon^3\right)
\end{cases}
$$

$$
\begin{cases}
\partial_t \lambda = \Omega + \chi \Pi_b + \mathcal{O}\left(\epsilon^3\right) \\
\partial_t \Omega = -3H_b\Pi_b\left(\chi + \kappa + \frac{\Omega}{\Pi_b}\right) + \mathcal{O}\left(\epsilon^3\right) \\
\partial_t h_{ij} = -2\tilde{A}_{ij} - \frac{2}{3}\eta_{ij}\overline{D}_k\beta^k + \overline{D}_j\beta_i + \overline{D}_i\beta_j + \beta^k\eta_{ij,k} + \mathcal{O}\left(\epsilon^3\right) \\
3\dot{\xi} - 3\beta^k\partial_k\zeta_b - 3H_b\chi - 3H_b\kappa - \overline{D}_i\beta^i = \mathcal{O}\left(\epsilon^3\right) \\
\kappa\left(2 + \frac{\partial_t H_b}{H_b^2}\right) + \frac{1}{H_b}\dot{\kappa} + 3\chi + \frac{8\Omega}{H_b}\sqrt{\frac{\pi}{3}} = \mathcal{O}\left(\epsilon^3\right) \\
\partial_t \tilde{A}_{ij} + 3\frac{\dot{a}}{a}\tilde{A}_{ij} = a^{-2}e^{-2\zeta_b}\mathfrak{R}_{ij}\left(\zeta_b\right) + \mathcal{O}\left(\epsilon^4\right) \\
\overline{\triangle}\zeta_b + \frac{1}{2}\overline{D}^k\zeta_b\overline{D}_k\zeta_b = 4\pi a^2 e^{2\zeta_b}\Pi_b^2\left(\kappa - \frac{\Omega}{\Pi_b}\right) + \mathcal{O}\left(\epsilon^4\right) \\
\overline{D}_j\tilde{A}_i^j + 3\tilde{A}_i^j\overline{D}_j\zeta_b + \Pi_b\left(2\sqrt{\frac{4\pi}{3}}\overline{D}_i\kappa + 8\pi\partial_i\lambda\right) = \mathcal{O}\left(\epsilon^5\right)
\end{cases}
$$

$$
\mathfrak{R}_{ij}\left(\zeta_b\right) = \overline{D}_i\zeta_b\overline{D}_j\zeta_b + \frac{1}{3}\eta_{ij}\overline{\triangle}\zeta_b - \overline{D}_i\overline{D}_j\zeta_b - \frac{1}{3}\eta_{ij}\overline{D}^k\zeta_b\overline{D}_k\zeta_b
$$

$$
\overline{\triangle}\zeta_b = \frac{e^{\frac{\zeta_b}{2}}}{4}\left(\overline{D}^k\zeta_b\overline{D}_k\zeta_b + 2\overline{\triangle}\zeta_b\right)
$$

# Gradient expansion solution

$$\Omega = -\frac{3}{16a_0^2}\sqrt{\frac{3}{\pi}}e^{-2\zeta_b}\left(\overline{D}^k\zeta_b\overline{D}_k\zeta_b + 2\overline{\triangle}\zeta_b\right)t^{\frac{1}{3}}$$
$$\kappa = \frac{3}{8a_0^2}e^{-2\zeta_b}\left(\overline{D}^k\zeta_b\overline{D}_k\zeta_b + 2\overline{\triangle}\zeta_b\right)t^{\frac{4}{3}}$$
$$\chi = \frac{9}{8a_0^2}e^{-2\zeta_b}\left(\overline{D}^k\zeta_b\overline{D}_k\zeta_b + 2\overline{\triangle}\zeta_b\right)t^{\frac{4}{3}}$$
$$\lambda = 0$$
$$\hat{\xi} = \frac{3e^{-2\zeta_b}}{8a_0^2}\left(\overline{D}^k\zeta_b\overline{D}_k\zeta_b + 2\overline{\triangle}\zeta_b\right)t^{\frac{4}{3}}$$
$$\tilde{A}_{ij} = \frac{3}{4a_0^2}\zeta_{ij}\left(\zeta_b\right)e^{-2\zeta_b}t^{\frac{1}{3}}$$
$$h_{ij} = -\frac{9}{8a_0^2}\mathfrak{R}_{ij}\left(\zeta_b\right)e^{-2\zeta_b}t^{\frac{4}{3}}$$

$$\Omega = -\frac{1}{4a_0^2}\sqrt{\frac{3}{\pi}}e^{-2\zeta_b}\left(\overline{D}^k\zeta_b\overline{D}_k\zeta_b + 2\overline{\triangle}\zeta_b\right)t^{\frac{1}{3}}$$
$$\kappa = 0$$
$$\chi = \frac{2e^{-2\zeta_b}}{a_0^2}\left(\overline{D}^k\zeta_b\overline{D}_k\zeta_b + 2\overline{\triangle}\zeta_b\right)t^{\frac{4}{3}}$$
$$\lambda = \frac{1}{16a_0^2}\sqrt{\frac{3}{\pi}}e^{-2\zeta_b}\left(\overline{D}^k\zeta_b\overline{D}_k\zeta_b + 2\overline{\triangle}\zeta_b\right)t^{\frac{4}{3}}$$
$$\hat{\xi} = \frac{e^{-2\zeta_b}}{2a_0^2}\left(\overline{D}^k\zeta_b\overline{D}_k\zeta_b + 2\overline{\triangle}\zeta_b\right)t^{\frac{4}{3}}$$
$$\tilde{A}_{ij} = \frac{3}{4a_0^2}\mathfrak{R}_{ij}\left(\zeta_b\right)e^{-2\zeta_b}t^{\frac{1}{3}}$$
$$h_{ij} = -\frac{9}{8a_0^2}\mathfrak{R}_{ij}\left(\zeta_b\right)e^{-2\zeta_b}t^{\frac{4}{3}}$$