Revisiting Stochastic Inflation with Perturbation Theory

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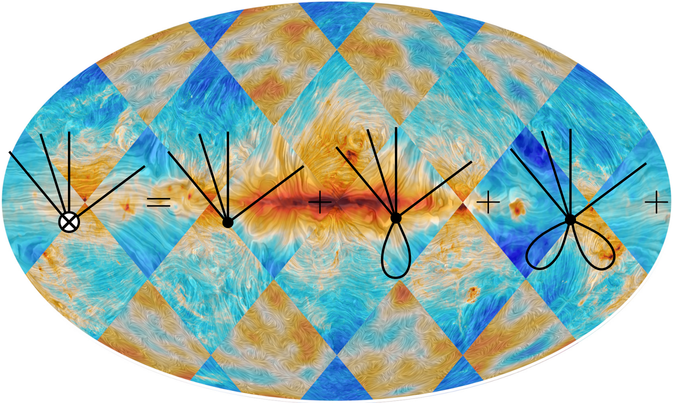


Put a scalar QFT on dS and just wait...

Related issues:

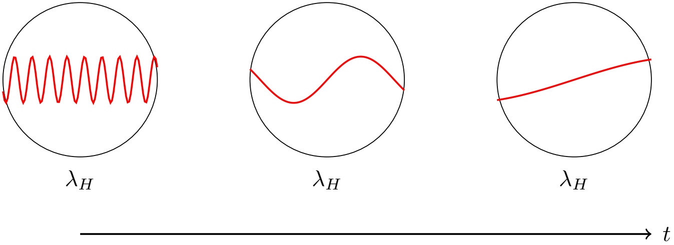
- dS stability
- cmb/lss phenomenology
- PT during inflation
- DE?

The problem

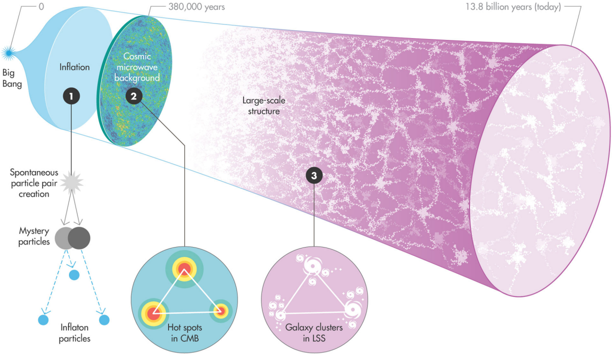
Probe scalar in the Poincare patch of (rigid) de Sitter space ($H =$ const):

$$\varphi,\ V(\varphi) \quad \Big| \quad ds^2 = a^2(\tau)(-dt^2 + dx^2),\ a = (H\tau)^{-1}.$$

Wavelengths stretch $\lambda = \ell a$, wavevectors shrink $p = k/a$.



- ⋆ Q: What is the statistics (correlators/PDF) of long modes? (random process in an open system)
- ⋆ Methods: QFT (Feynman graphs) = Stochastic (Langevin eq)



$$\boxed{\mathrm{P}(\varphi) \rightarrow \mathrm{P}(\delta T) \rightarrow \mathrm{P}(\delta_{\mathrm{g}})\ ,\ \mathrm{P}(\delta_{\mathrm{pbh}})}$$

$\mathcal{V}(\varphi)$ on dS

$$S = \int \mathrm{d}^3x\,\mathrm{d}\tau\, a^2 \left[\frac{1}{2}\dot{\varphi}^2 - \frac{(\nabla\varphi)^2}{2} - a^2\mathcal{V}(\varphi)\right]$$

$$\mathcal{V}(\varphi) = \sum_{n=2}^{\infty} \frac{\lambda_n^{\mathrm{bare}}}{n!}\varphi^n$$

We will be interested in single-point statistics (i.e. histograms marginalised over $\boldsymbol{x}$)

$$\langle\varphi^n\rangle = \langle\varphi(\boldsymbol{x}_1)\cdots\varphi(\boldsymbol{x}_n)\rangle\Big|_{\boldsymbol{x}_1=\cdots=\boldsymbol{x}_n} \propto \lambda_n^{\mathrm{obs}}$$

Edgeworth expansion of the PDF truncated to linear order in the interactions (single-vertex diagrams):

$$\rho(\varphi, t) = \frac{e^{-\frac{1}{2}\frac{\varphi^2}{\sigma^2(t)}}}{\sqrt{2\pi\sigma^2(t)}}\left[1 + \Delta(\varphi, t)\right]$$

with

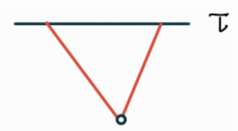
$$\Delta(\varphi, t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\left\langle \varphi^n(t) \right\rangle_c}{\sigma^n} \mathrm{He}_n(\varphi/\sigma)$$

$$\boxed{\text{Perturbative QFT} = \text{Gaussian variables} =}$$

$$\boxed{\text{Feynman rules} = \text{Wick's theorem}}$$

UV/IR structure of correlators

The simplest correlator is the (free theory) one-point variance:



$$\sigma^2 = H^2 \int_0^\infty \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3} \frac{1}{k^3} \left(1 + (k\tau)^2\right)$$

or, in physical momentum $p \equiv Hk\tau$

$$\sigma^2 = \left(\frac{H}{2\pi}\right)^2 \int_0^\infty \frac{\mathrm{d}p}{p} \left(1 + \frac{p^2}{H^2}\right)$$

UV divergence: $\propto p^2$

IR divergence: $\propto \ln p$

Cure for external legs: restrict to long modes

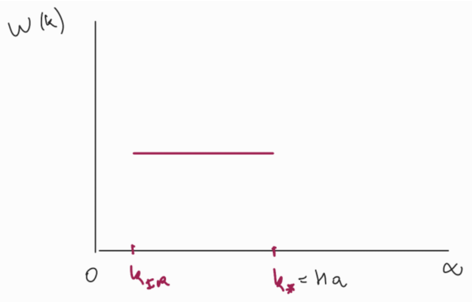
Long-mode statistics

Long-short split: $\varphi = \varphi_S + \varphi_L$:

$$\varphi_L(x,t) \equiv \hat{\boldsymbol{L}}\left\{\varphi(x,t)\right\} = \int_k e^{ikx} W(k)\tilde{\varphi}_k(t)$$

$$W(k) = \theta(Ha - k) \times \theta(k - k_*(t_{\rm i}))$$

$\boxed{\text{Cut off external legs at } k_*(t_{\rm i}) \equiv k_{\rm IR}\text{. This is our choice!}}$



Secular structure of correlators

Now

$$\sigma_L^2(t) = \left(\frac{H}{2\pi}\right)^2 \int_{H/a}^{H} \frac{\mathrm{d}p}{p} = \frac{H^3 t}{4\pi^2}$$

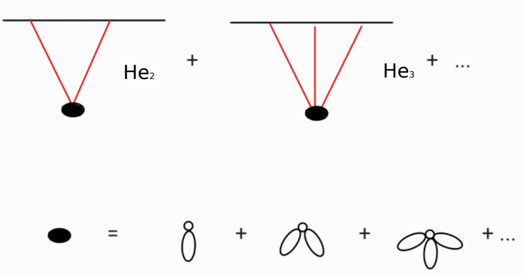
We regularized but we now have a secular behaviour.

Linear modes independent $\to$ variance additive $\to$ blows up asymptotically (secular).

$\boxed{\text{Cure: resummation}}$ (e.g. stochastic formalism)

What does the Fokker-Planck resum?

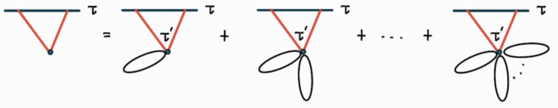
FP resums the Edgeworth expansion of the PDF with loop-corrected correlators. For a single-vertex:



The loop time-dependence matters!

Daisy loops

Do loops change the secular growth?



$$\mathcal{V}(\varphi) = \lambda_4^{\rm bare}\varphi^4 + \lambda_6^{\rm bare}\varphi^6 + \cdots\,, \quad (\text{e.g. axion } \mathcal{V} \propto 1 - \cos\varphi/f)$$

Recall that loops are virtual processes; they control the validity.

Cure for internal legs: UV & IR cutoffs or dim reg $+$ renormalization

Now there is a choice to be made! Comoving vs physical IR cutoff.

Is $t_{\rm i}$ part of the theory?

Let us first choose a physical cutoff (dS invariant regulator). Then

$$\sigma_0^2 = \left(\frac{H}{2\pi}\right)^2 \int_{\Lambda_{\mathrm{IR}}}^{\Lambda_{\mathrm{UV}}} \frac{\mathrm{d}p}{p} = \left(\frac{H}{2\pi}\right)^2 \ln \frac{\Lambda_{\mathrm{UV}}}{\Lambda_{\mathrm{IR}}}$$

SK, in-in, WFU: n-pt functions loop-resummed to:

$$\left\langle \varphi^n(\tau) \right\rangle_c = -\frac{8\pi}{H^4} \mathrm{Im}\left\{ \int_0^\infty \mathrm{d}x \int_{\tau_{\mathrm{i}}}^{\tau} \mathrm{d}\bar{\tau} \, \frac{x^2}{\bar{\tau}^4} \lambda_n^{\mathrm{obs}} \left[g(x, \bar{\tau}, \tau) \right]^n \right\}$$

with g the propagator and

$$\lambda_n^{\mathrm{obs}} = \sum_{L=0} \frac{\lambda_{n+2L}^{\mathrm{bare}}}{L!} \left(\frac{\sigma_0^2}{2}\right)^L$$

(This is nothing but the coefficients of a Weierstrass transform)

The PDF of long modes

Edgeworth expansion of the PDF truncated to single-vertex diagrams:

$$\rho(\varphi, t) = \frac{e^{-\frac{1}{2}\frac{\varphi^2}{\sigma^2(t)}}}{\sqrt{2\pi\sigma^2(t)}}\left[1 + \Delta(\varphi, t)\right]$$

with

$$\Delta(\varphi, t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\left\langle \varphi^n(t) \right\rangle_c}{\sigma^n} \mathrm{He}_n(\varphi/\sigma)$$

Upon using

$$\mathcal{O}_\varphi \mathrm{He}_n(\varphi/\sigma) = -n\mathrm{He}_n(\varphi/\sigma)$$

the PDF can be resummed:

$$\Delta(\varphi,\tau) = \frac{8\pi}{H^4}\mathrm{Im}\int_0^\infty \mathrm{d}x \int_{\tau_\mathrm{i}}^{\tau} \mathrm{d}\bar{\tau}\, \frac{x^2}{\bar{\tau}^4}\left(\frac{g(x,\bar{\tau},\tau)}{\sigma^2}\right)^{-\mathcal{O}_\varphi} \underbrace{e^{\frac{\sigma_0^2-\sigma_L^2(t)}{2}\frac{\partial^2}{\partial\varphi^2}}\mathcal{V}(\varphi)}_{\mathcal{V}_\mathrm{ren}(\varphi)}$$

such that $\rho = \rho_G\left[1+\Delta\right] = \rho_G + \Delta_\mathrm{NG}$.

In the $k\tau \to 0$ limit this simplifies considerably:

$$\boxed{\Delta_\mathrm{NG} = \frac{4\pi^2}{3H^4}\rho_G\,\mathcal{O}_\varphi\mathcal{V}_\mathrm{ren}} \quad \text{and} \quad \boxed{\rho = \rho_G + \frac{4\pi^2}{3H^4}\rho_G\,\mathcal{O}_\varphi\mathcal{V}_\mathrm{ren}}$$

This PDF respects a **Gaussian** IC

The Fokker-Planck

Now write $\rho = \rho_G + \Delta_{\rm NG}$ and use the fact that

$$\frac{\partial}{\partial N}\Big(\rho - \Delta_{\rm NG}\Big) = \frac{1}{2}\frac{\partial \sigma^2}{\partial N}\,\frac{\partial^2}{\partial \varphi^2}\Big(\rho - \Delta_{\rm NG}\Big)$$

Upon massaging, a physical loop cutoff leads to an FP equation:

$$\boxed{\dot{\rho} = \frac{H^3}{8\pi^2}\frac{\partial^2}{\partial\varphi^2}\left[\rho\left(1 - \frac{2t}{3H}\mathcal{V}''_{\rm ren}\right)\right] + \frac{1}{3H}\frac{\partial}{\partial\varphi}\left(\rho\mathcal{V}'_{\rm ren}\right)}$$

Equilibrium?

$$\dot{\rho} = \frac{H^3}{8\pi^2}\frac{\partial^2}{\partial\varphi^2}\left[\rho\left(1 - \frac{2t}{3H}\mathcal{V}''_{\rm ren} + \mathcal{O}\left(t^2\mathcal{V}^2\right)\right)\right] + \frac{1}{3H}\frac{\partial}{\partial\varphi}\left(\rho\mathcal{V}'_{\rm ren}\right)$$

Langevin equation

The Langevin equation is just the interaction picture field:

$$\varphi(x,t) = U(t)\varphi_I(x,t)U^\dagger(t)$$

To first order in the potential, this becomes

$$\varphi \simeq \varphi_I - \frac{1}{3H}\int^t \mathrm{d}t'\, \frac{\partial \mathcal{V}}{\partial \varphi}\left[\varphi_I(t')\right]$$

and restricting to long modes via the W operator:

$$\varphi_L(t) \simeq \varphi_G(t) - \frac{1}{3H}\int_{t_\mathrm{i}}^t \mathrm{d}t'\, W\left\{\frac{\partial \mathcal{V}}{\partial \varphi}\left[\varphi_I(t')\right]\right\}$$

Now I can compute $\langle \varphi_L^n \rangle$ using Wick contractions. This leads to the same FP.

When we linearise the KG in the short modes, we get something slightly different

$$\varphi_L(t) \simeq \varphi_G(t) - \frac{1}{3H}\int_{t_\mathrm{i}}^{t} \mathrm{d}t' \, \frac{\partial \mathcal{V}}{\partial \varphi} \left[W\{\varphi_l(t')\} \right]$$

This corresponds to a comoving IR cutoff.
We can use this to compute correlators (Wick contractions):

$$\left\langle \varphi^n(t) \right\rangle_c = -\frac{4\pi^2 n}{3H^4}\sigma^{2n}(t)\sum_L^{\infty} \frac{\lambda_{n+2L}}{(n+L)L!}\left(\frac{\sigma^2(t)}{2}\right)^L .$$

Upon resumming the Edgeworth series, this leads to a PDF reading

$$\rho_1 = \rho_G + \frac{4\pi^2}{3H^4}\rho_G \int_0^{\sigma^2} \frac{dy}{y}\left(\frac{\sqrt{y}}{\sigma}\right)^{-\mathcal{O}_\varphi} \mathcal{O}_\varphi \mathcal{V}\left(\varphi \frac{\sqrt{y}}{\sigma}\right) .$$

This respects

$$\boxed{\dot{\rho}_1 = \frac{H^3}{8\pi^2}\frac{\partial^2 \rho_1}{\partial \varphi^2} + \frac{1}{3H}\frac{\partial}{\partial \varphi}\left(\rho_0 \mathcal{V}'\right)}$$ with a Gaussian IC

To sum-up

- ⋆ resummation of Feynman diagrams = stochastic formalism with Gaussian IC
- ⋆ FP solutions from QFT
- ⋆ IR cutoff seems to matter (?)
- ⋆ Physical IR → renormalized potential (Arttu's talk)