Numerical tools for strong lensing

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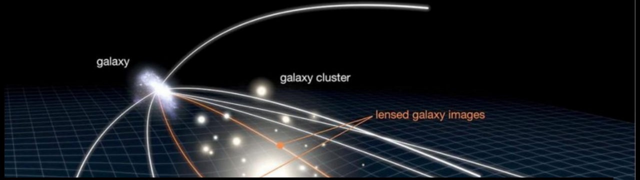
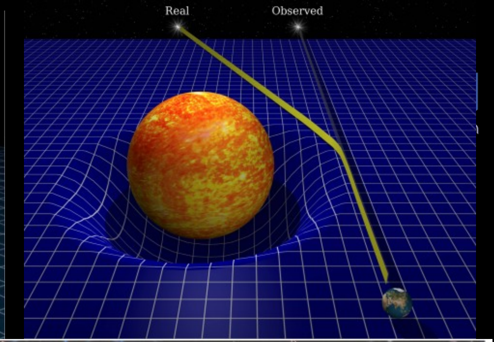
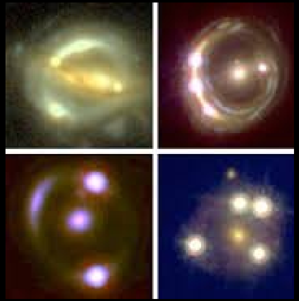
Introduction





- Light deflection
- Wide varie



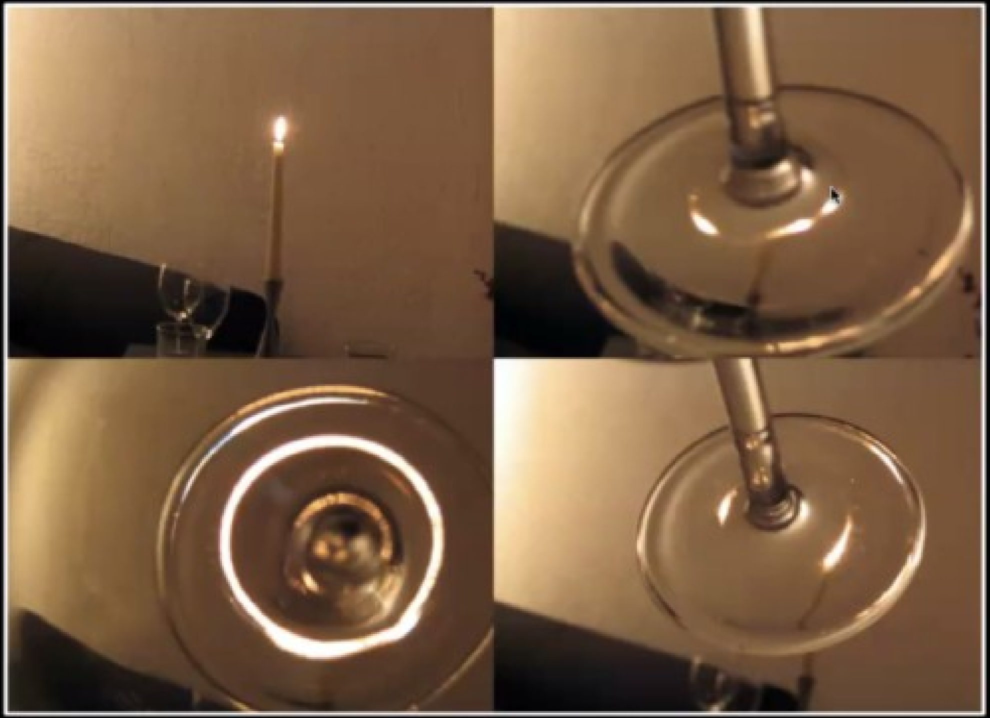


Image Credit: P. J. Marshall

- When impa
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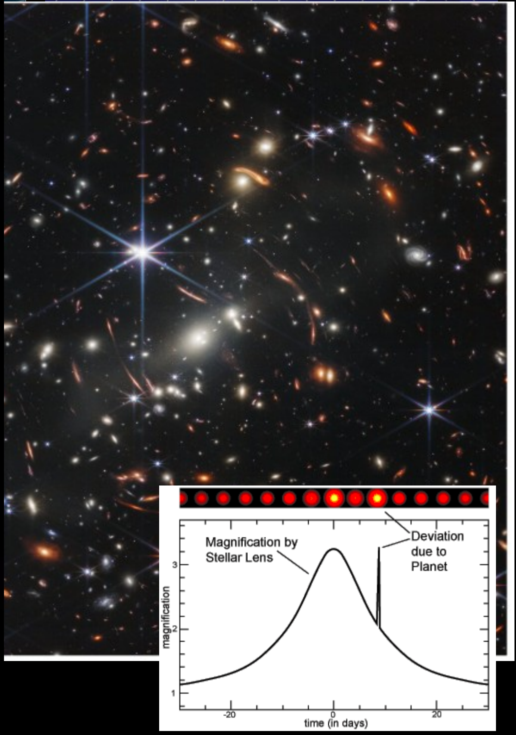
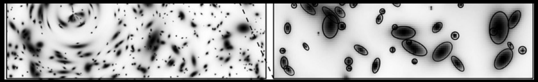




The lens equation for a thin lens

- Main hypotheses: thin, $\Psi \ll c^2$, stationnary $d\Psi/dt \sim 0$, FLRW elsewhere, transparent otherwise

$$\hat{\alpha} = \frac{2}{c^2} \int \nabla_\perp \Phi \, \mathrm{d}l$$

- Previous results if

$$\Phi(\vec{r}) = -GM/|\vec{r}|$$

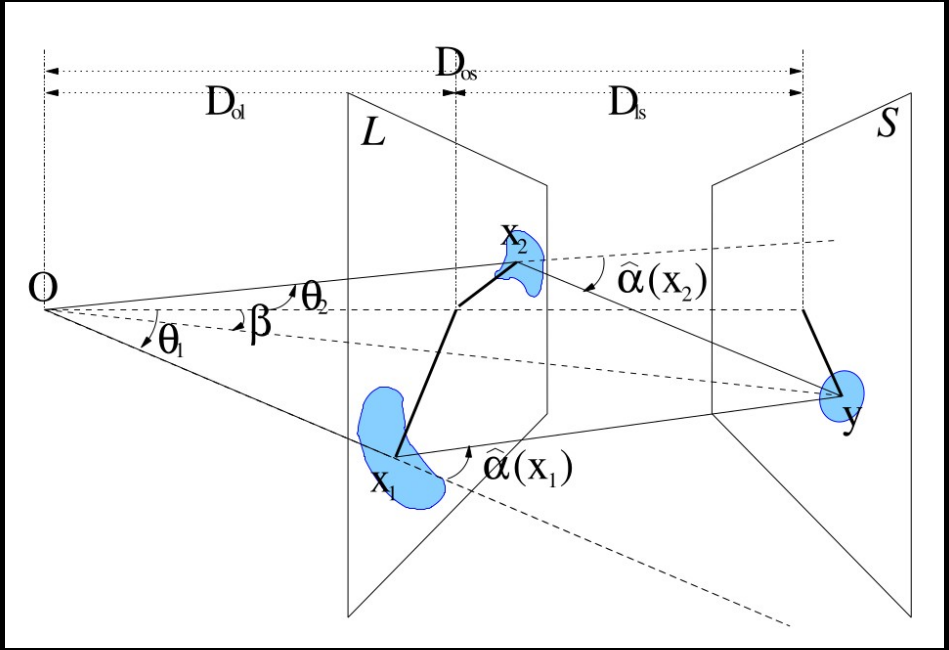
$$\hat{\alpha} = \frac{4GM}{c^2} \frac{\vec{x}}{|\vec{x}|^2},$$

- Geometry yields

$$D_{\rm os}\beta + D_{\rm ls}\hat{\alpha} = D_{\rm os}\theta.$$

- Introduction reduced deflection angle

$$\alpha = \frac{D_{\rm ls}}{D_{\rm os}} \hat{\alpha}$$



The main lens equation

$$\vec{\beta} = \vec{\theta} - \vec{\alpha} \equiv \vec{\theta} - \vec{\nabla}\psi(\vec{\theta})$$

Potential, convergence, arrival time

- where $\psi(\vec{\theta}) = \frac{2}{c^2}\frac{D_{\rm ls}D_{\rm ol}}{D_{\rm os}}\phi(\vec{\theta})$ and $\phi = \int \Phi \, \mathrm{d}z$

The lens potential $\Delta\psi = 2\kappa = \vec{\nabla}.\vec{\alpha}$

- Equivalent of $\Delta\phi(\vec{\theta}) = 4\pi G\Sigma(\vec{\theta})$

With convergence $\kappa \equiv \Sigma/\Sigma_{\rm crit}$ is surface density normalised by critical density

$$\Sigma_{\rm crit} = \frac{c^2}{4\pi G}\frac{D_{\rm os}}{D_{\rm ol}D_{\rm ls}}$$

- Relations

$$\vec{\alpha}(\vec{\theta}) = \frac{1}{\pi}\int_{\mathbb{R}^2} \mathrm{d}^2\vartheta \; \kappa(\vec{\vartheta})\frac{\vec{\theta}-\vec{\vartheta}}{|\vec{\theta}-\vec{\vartheta}|^2},$$

$$\psi(\vec{\theta}) = \frac{1}{\pi}\int_{\mathbb{R}^2} \mathrm{d}^2\vartheta \; \kappa(\vec{\vartheta}) \ln|\vec{\theta}-\vec{\vartheta}|.$$

Arrival time

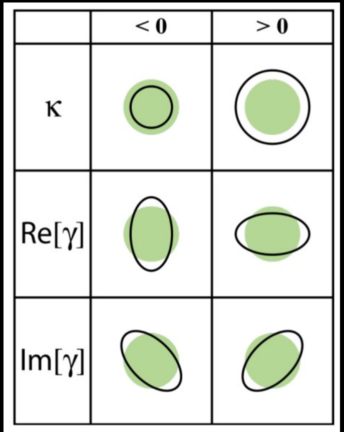
$$ct = \frac{D_{\rm ol}D_{\rm ls}}{D_{\rm os}}\left((\vec{\theta}-\vec{\beta})^2 - \psi(\vec{\theta})\right)$$

Fermat principle $\nabla_{\vec{\theta}}t = 0$ -> lens eq.

Local distorsions: magnification, shear...

- The lens equation... ‘’inverse’’ mapping

$$\vec{\beta} = \vec{\theta} - \vec{\alpha} \equiv \vec{\theta} - \vec{\nabla}\psi(\vec{\theta})$$



- Locally:

$$\mathcal{A}_{ij}(\vec{\theta}) = \frac{\partial\vec{\beta}}{\partial\vec{\theta}} = (\delta_{ij} - \psi_{,ij}) \equiv \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix}$$

- Convergence $\varkappa$ (isotropic distorsions) shear γ (anisotropic distorsions)

$$\kappa = \frac{1}{2}(\psi_{,11} + \psi_{,22})$$
$$\gamma_1 = \frac{1}{2}(\psi_{,11} - \psi_{,22})$$
$$\gamma_2 = \psi_{,12}$$

Jacobian (magnification):

$$\mu^{-1} = \det\mathcal{A} = (1-\kappa)^2 - |\gamma|^2$$

Local distorsions: magnification, shear…

- Shape of images : surface brightness conservation

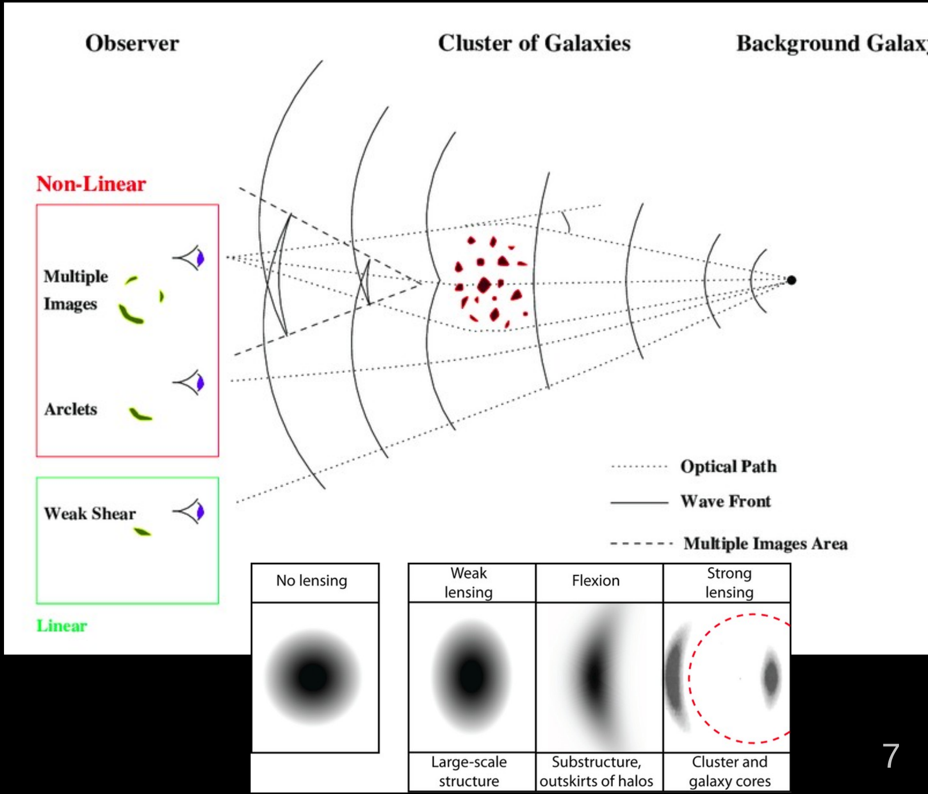
$$I(\vec{\theta}) = I_s(\vec{\theta} - \vec{\nabla}(\vec{\theta}))$$

- Locally:

$$I(\vec{\theta}) = I_s\left[\vec{\beta}_0 - \mathcal{A}(\vec{\theta}_0)(\vec{\theta} - \vec{\theta}_0)\right]$$

- Convergence $\varkappa$ (isotropic distorsions)
- shear γ (anisotropic distorsions)

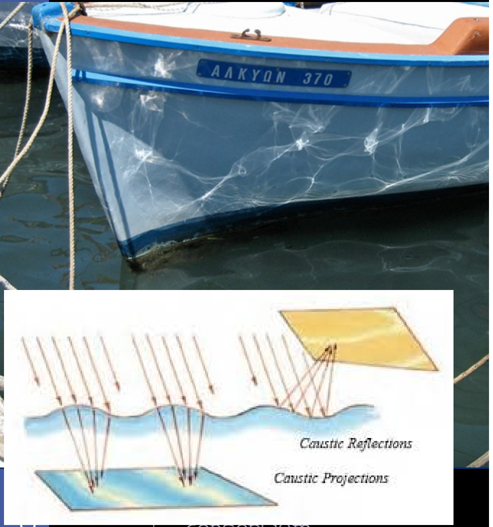
$\longrightarrow$ WEAK LENSING regime

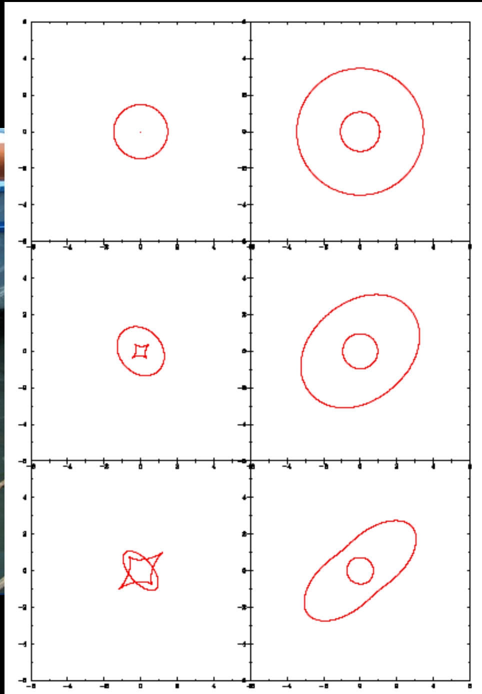


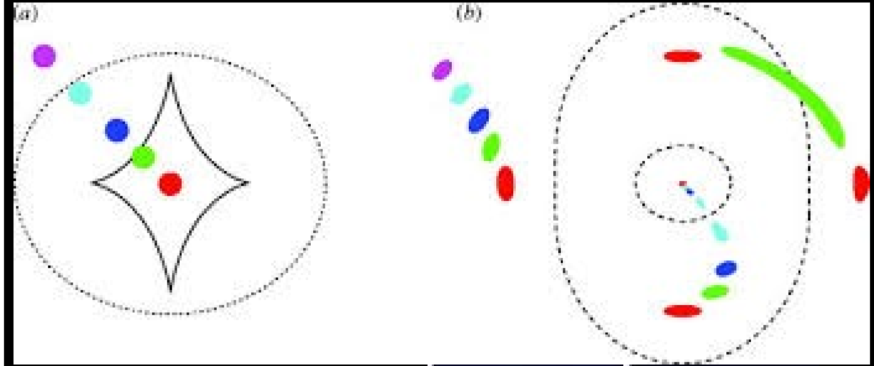
Critical/Caustic lines, image multiplicity

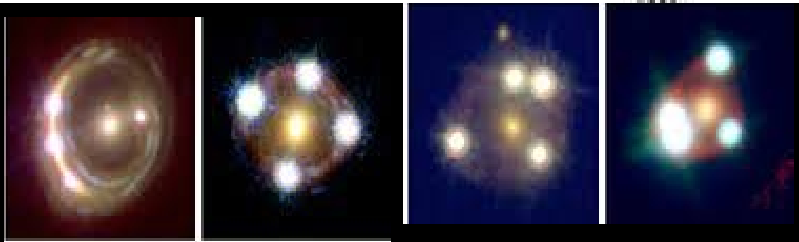
Critical lines : where magnification is infinite in image plane
Einstein radius Rein : Radius of the main (tangential critical line)

Caustic lines: image of critical lines through lens equation in the source plane









Inverting the lens equation

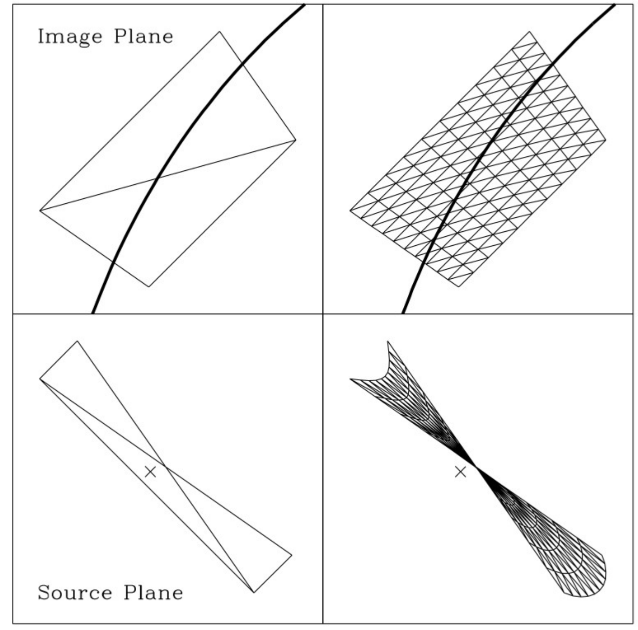
- Lens equation explicitly yields β knowing θ.

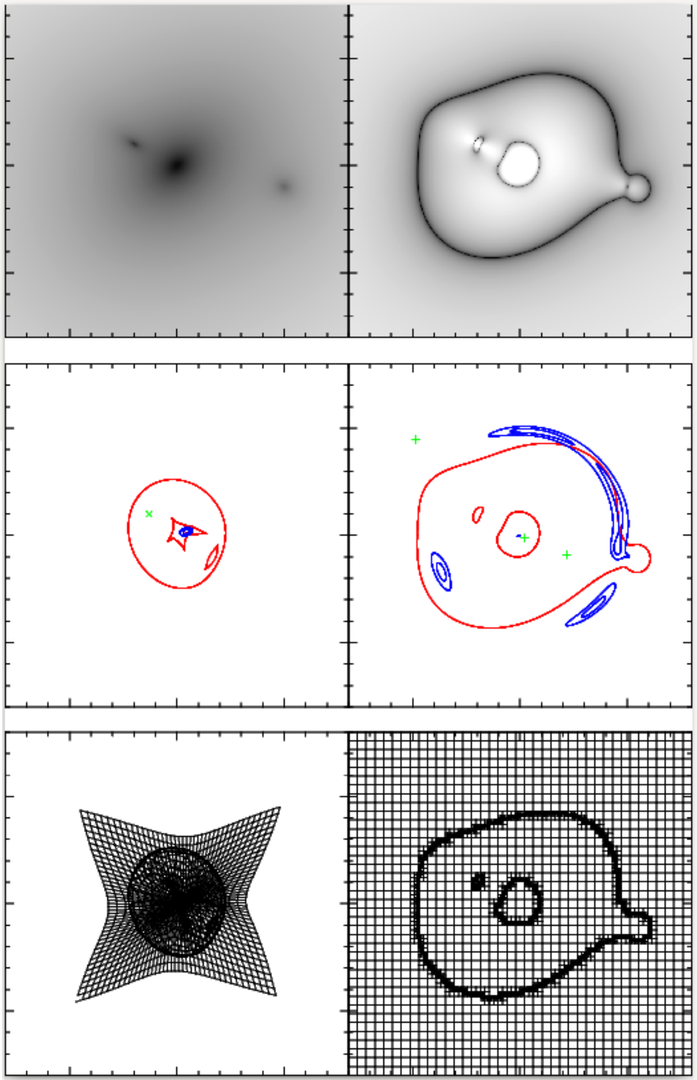
$$\vec{\beta} = \vec{\theta} - \vec{\alpha} \equiv \vec{\theta} - \vec{\nabla}\psi(\vec{\theta})$$

- How to go the othe
get **all** the θs, know
(important for mod

Numerical adaptiv

Same technique fo
lines and isophotes c





Degeneracies : MST

- Any transformation like this
  leaves lensing **observables** unchanged if unknown source plane is rescaled like
  (Same critical lines,... same mass in Rein!)

- Total magnification scales like
- Arrival times scale like
- Unless intrinsic source size/flux is known, mass sheet transformation (and more complex analogous) prevents robust density profile estimates

- Additional non-lensing data is needed to lift degeneracies (especially H_0 estimates)

Degeneracies : non-axisymmetry

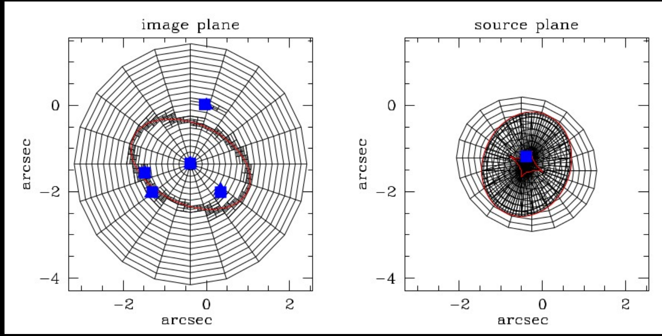
- Apart from monopole (isotropic matter distrib.), multipole expansion shows that m=2 and beyond terms at radius r come from both mass interior (eg internal ellipticity) and exterior (eg external shear) matter.

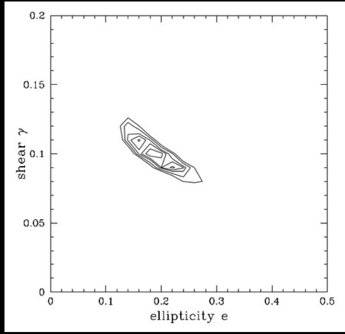
- Some degeneracy is expected since
  $R \sim R_{ein} \pm \delta$

$$\phi(x, y) = \phi_0(r) + \sum_{m=1}^{\infty} [\phi_{cm}(r) \cos m\theta + \phi_{sm}(r) \sin m\theta]$$

$$\psi_m(r) = -\frac{2\pi G}{m}\left[r^{-m}\int_0^r dx\, \Sigma_m(x) x^{m+1} + r^m \int_r^{\infty} dx\, \Sigma_m(x) x^{1-m}\right]$$

$$\psi_0(r) = 4\pi G\left[\log r \int_0^r dx\, \Sigma_m(x) x + \int_r^{\infty} dx\, \Sigma_m(x) x \log x\right]$$

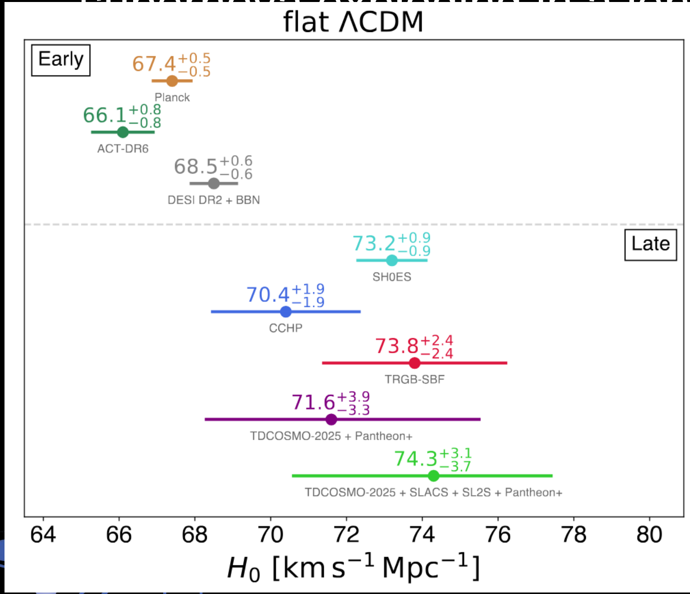


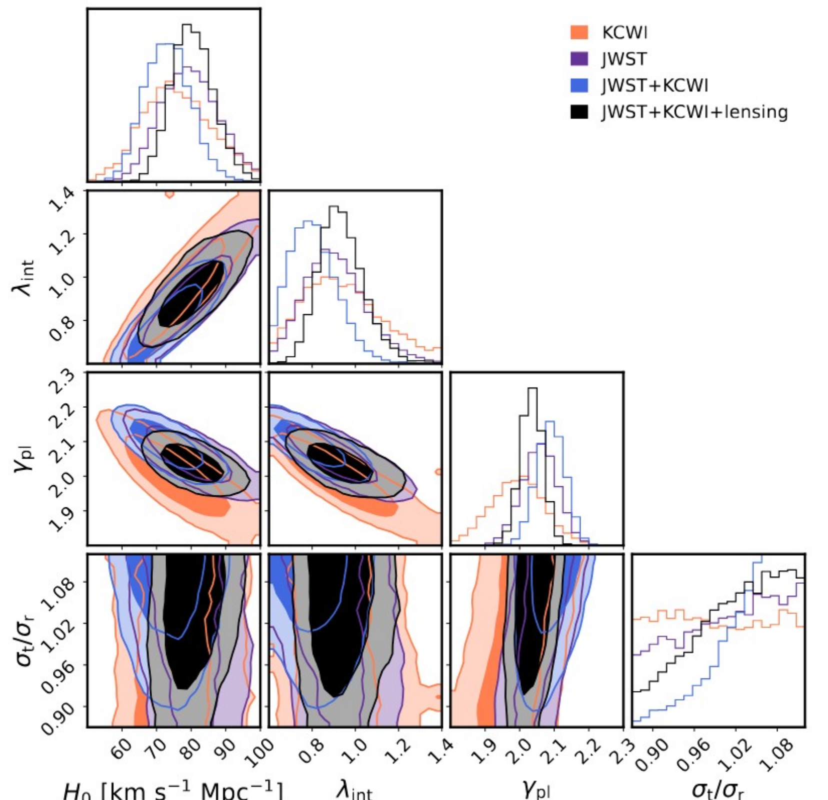


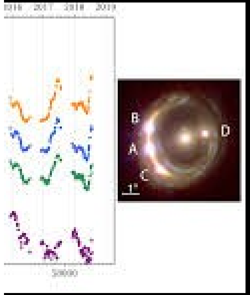
Time delays

$$ct = \frac{D_{\rm ol} D_{\rm ls}}{D_{\rm os}} \left((\vec{\theta} - \vec{\beta})^2 - \psi(\vec{\theta}) \right)$$

- Time delay between two paths $t_A - t_B \sim H_0^{-1}$ (Refsdal 64)
- Requires the monitoring of vari… a supernova exploding in a lense…

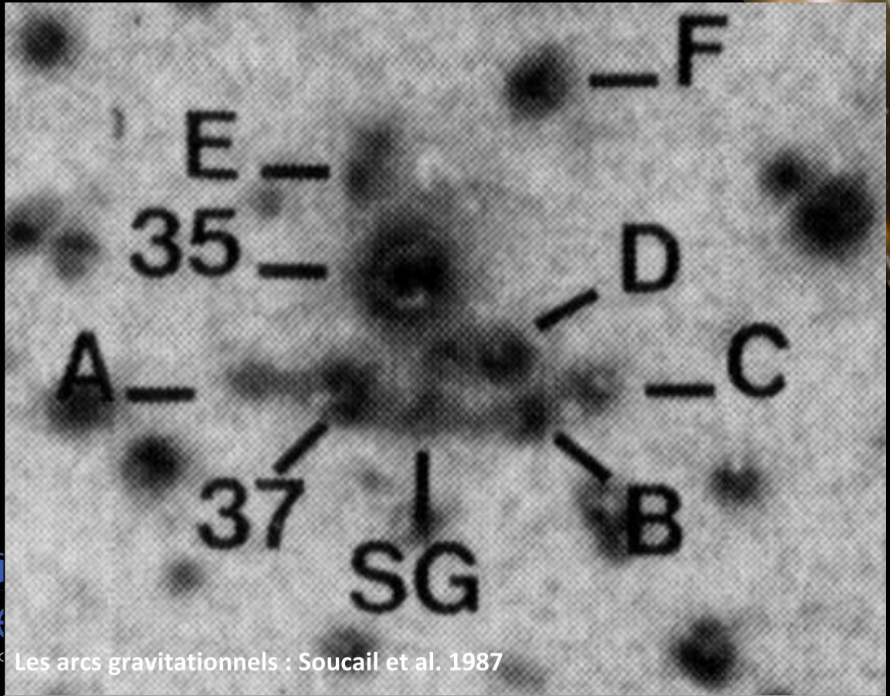




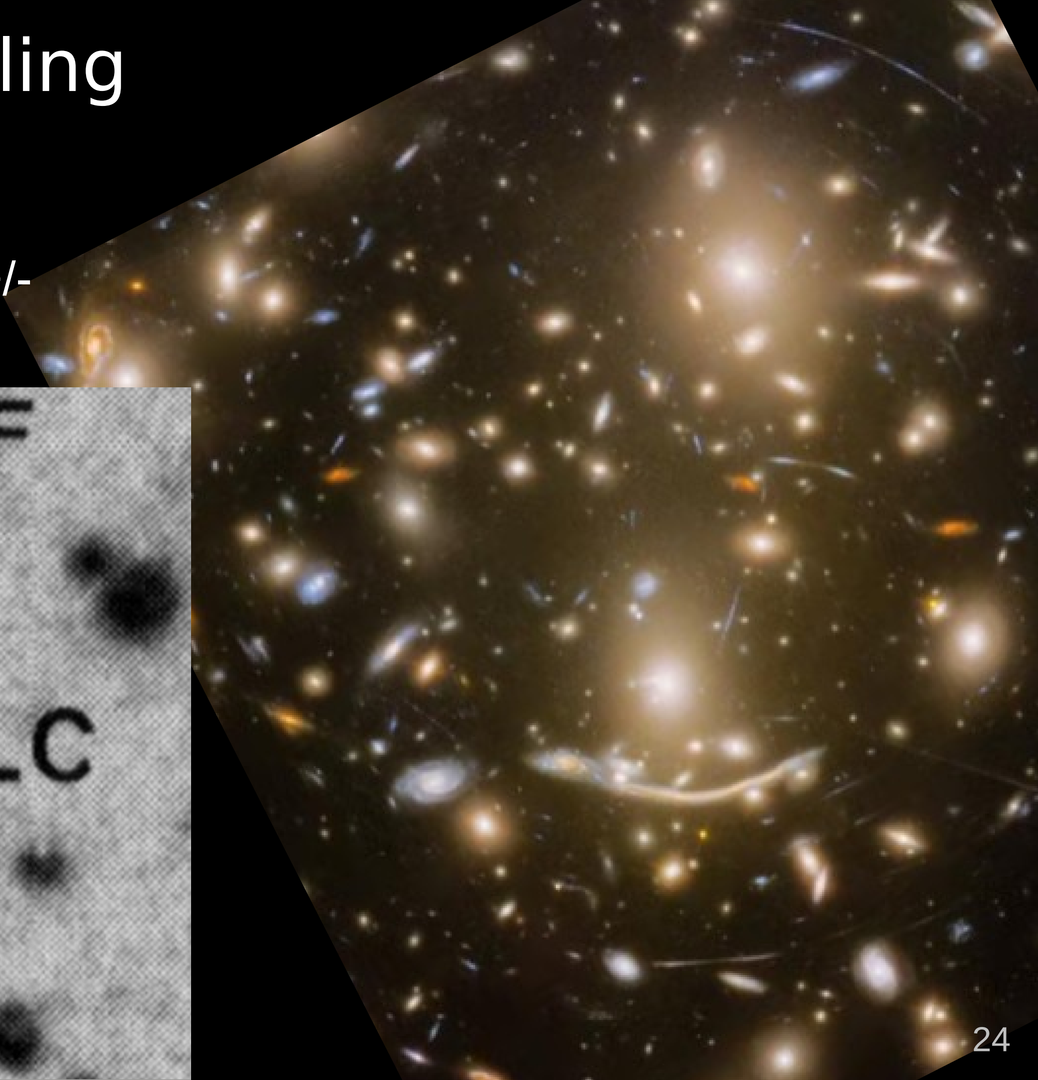


Cluster-scale SL modeling

- Space vs ground A370
- Sources petites --> fit positions +/-
- Source plane vs image plane
-



Les arcs gravitationnels : Soucail et al. 1987



Cluster-scale SL modeling

- Several sources at different redshift mixed with several fg galaxies : colour+resolution (morphology) crucial!
- Sources are relatively small compared to typical scale of potential variations. Point-source minimisation is ok!

$$\chi^2_{\mathrm{src},0} = \sum_{i=1}^{N-1} \frac{\left(\vec{\theta}_i - \vec{\nabla}\psi(\vec{\theta}_i) - \vec{\theta}_{i+1} + \vec{\nabla}\psi(\vec{\theta}_{i+1})\right)^2}{\sigma^2_{x,i}}$$

$$\begin{aligned} \vec{\beta} &= \vec{\theta}_1 - \vec{\nabla}\psi(\vec{\theta}_1) \\ &\vdots \\ &= \vec{\theta}_N - \vec{\nabla}\psi(\vec{\theta}_N) \end{aligned}$$

Fast but inaccurate

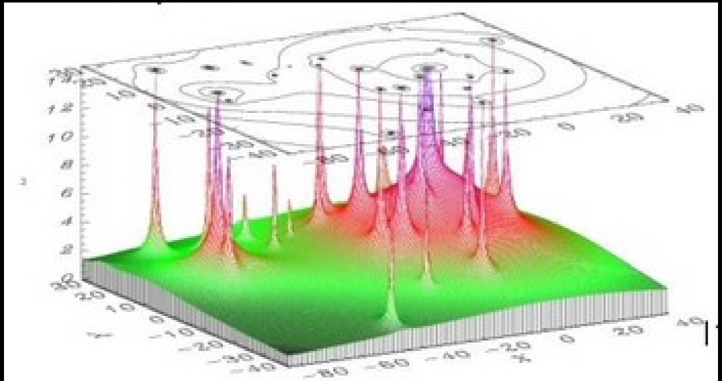
$$\chi^2_{\mathrm{img}} = \sum_{i=1}^{N} \delta\vec{\theta}_i^T \, \mathcal{C}_i^{-1} \, \delta\vec{\theta}_i$$

$$\delta\vec{\theta} = \vec{\theta}_{\mathrm{obs},i} - \vec{\theta}_{\mathrm{mod},i}$$

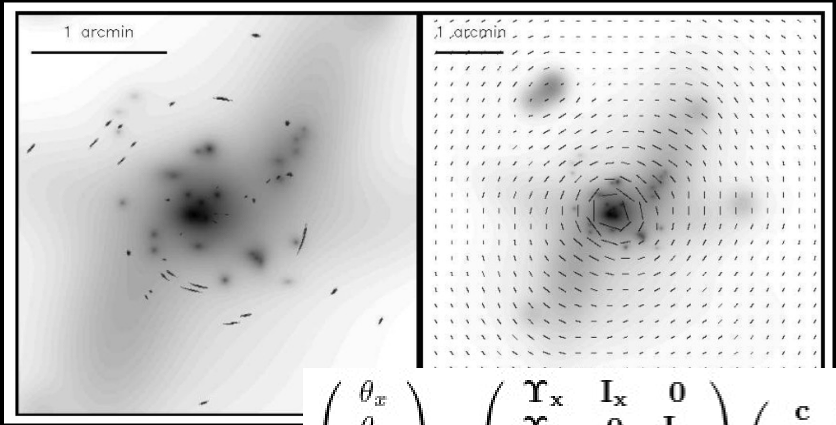
Requires inverting lens equation. Tedious !
Image plane rms often > image resolution !

Cluster-scale SL modeling

- Parametric vs grid-based (free form) models (and hybrid)



MCMC sampling of high-dim non-linear parameter space.
Slow but robust !
lenstool, glafics, glee, sl_fit, gravity.jl



$$\begin{pmatrix} \theta_x \\ \theta_y \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \Upsilon_x & I_x & 0 \\ \Upsilon_y & 0 & I_y \\ \Delta_1 & 0 & 0 \\ \Delta_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} c \\ \beta_x \\ \beta_y \end{pmatrix}$$

''Linear'' inversion, fast but unstable
Requires regularisation...
WSLAP, pixelens, SWUnited...

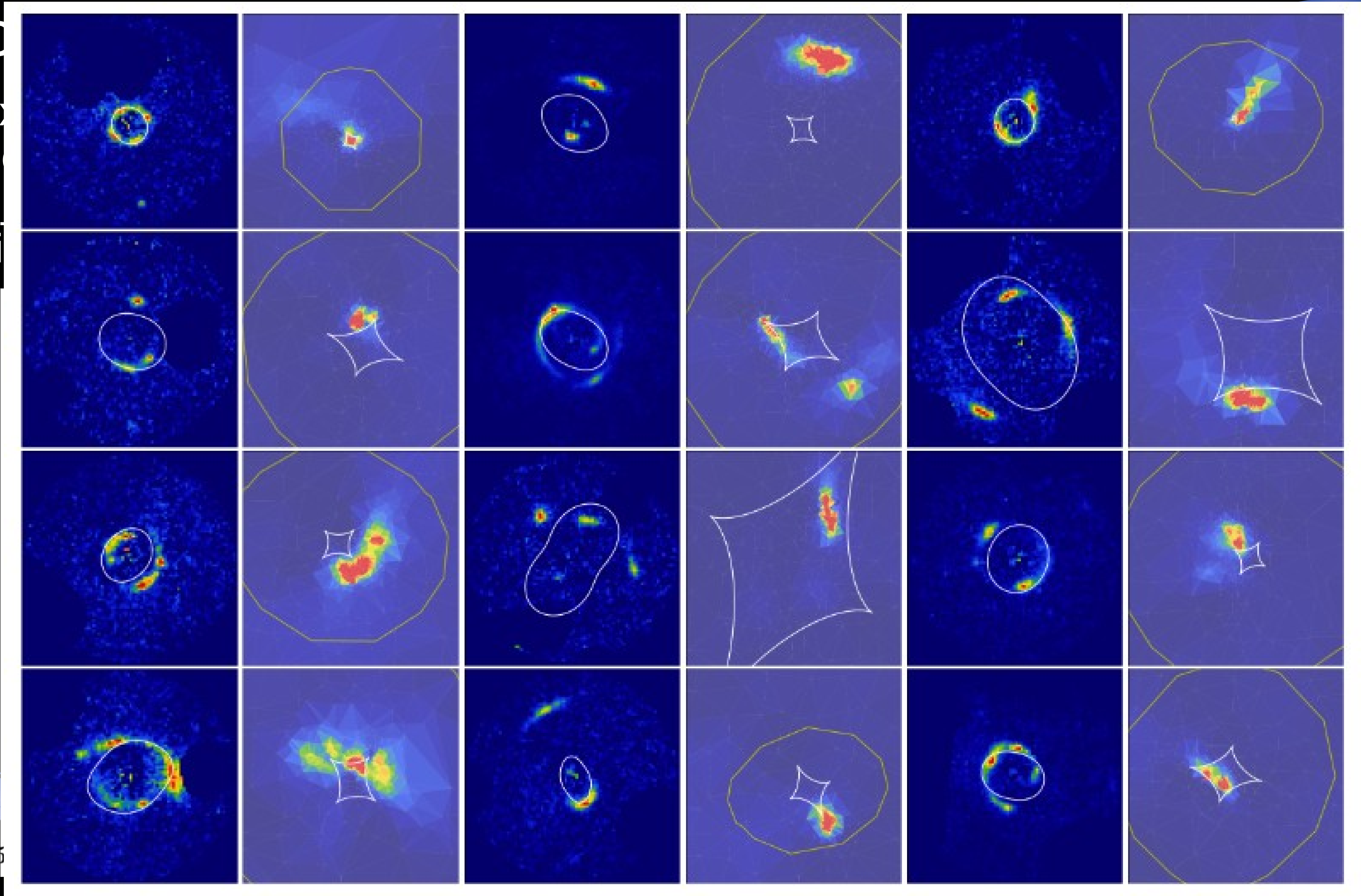
Cluster-scale SL modeling

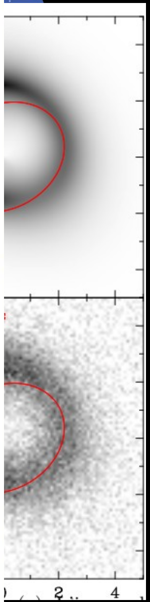
- Results:
  - HFF sims and challenge...
  - SN Refsdal, time delays on first lensed SN in MACSJ1149
  - Recent JWST models
  - Addition of kinematics, WL

See Marceau’s presentation

Mo

- Ex
- C
  sl
- Li

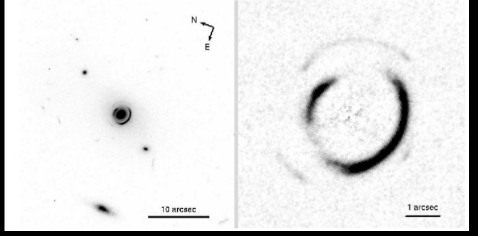


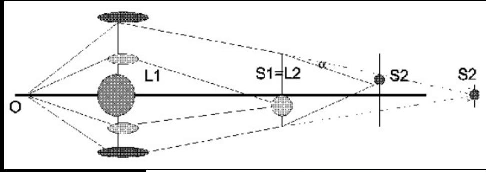


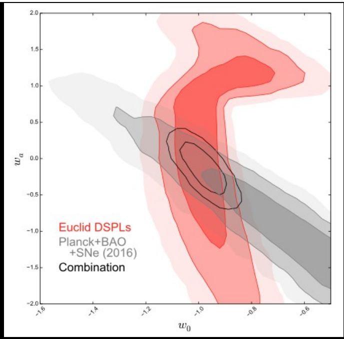
$HS\|^2$

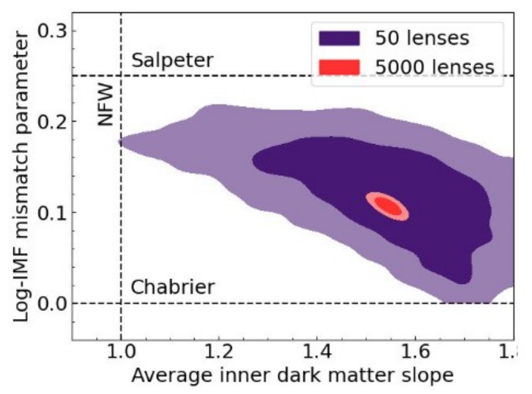
Modeling galaxy-scale lenses

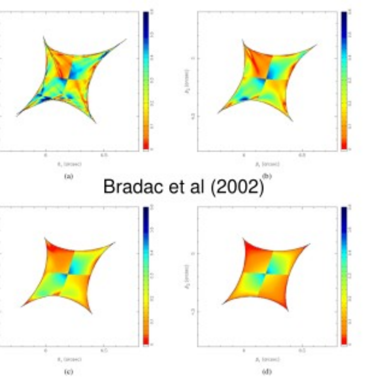
- Sous-structures ... multipoles
- Results
- DSPLs
- L+D modeling
- S+W in galaxies

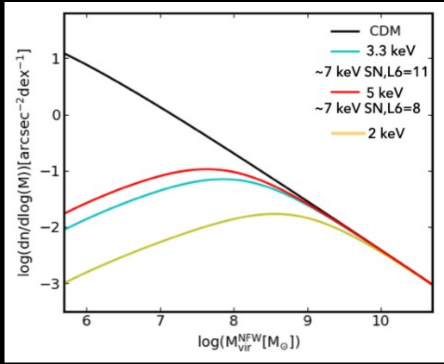






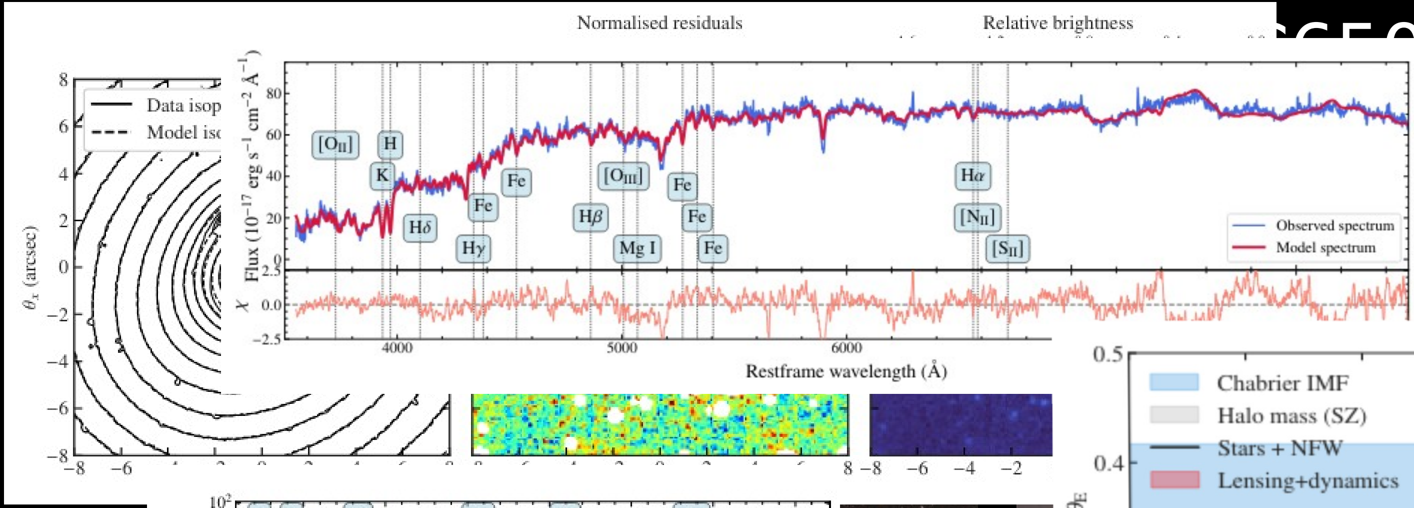


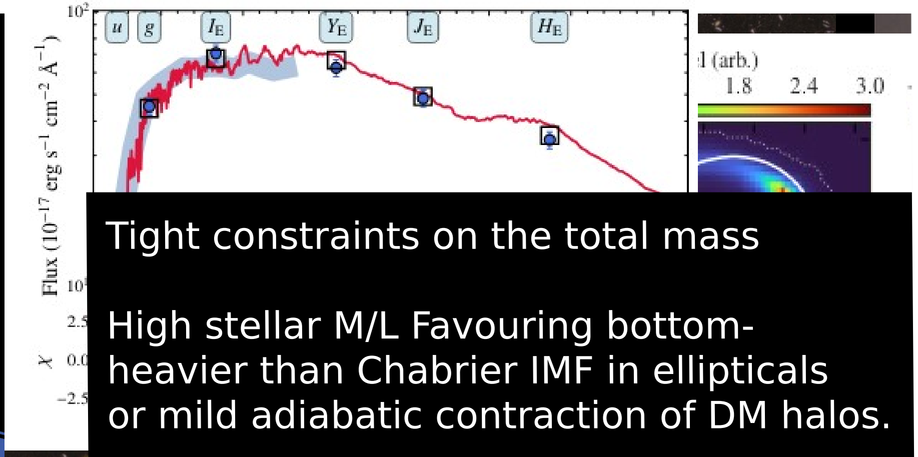


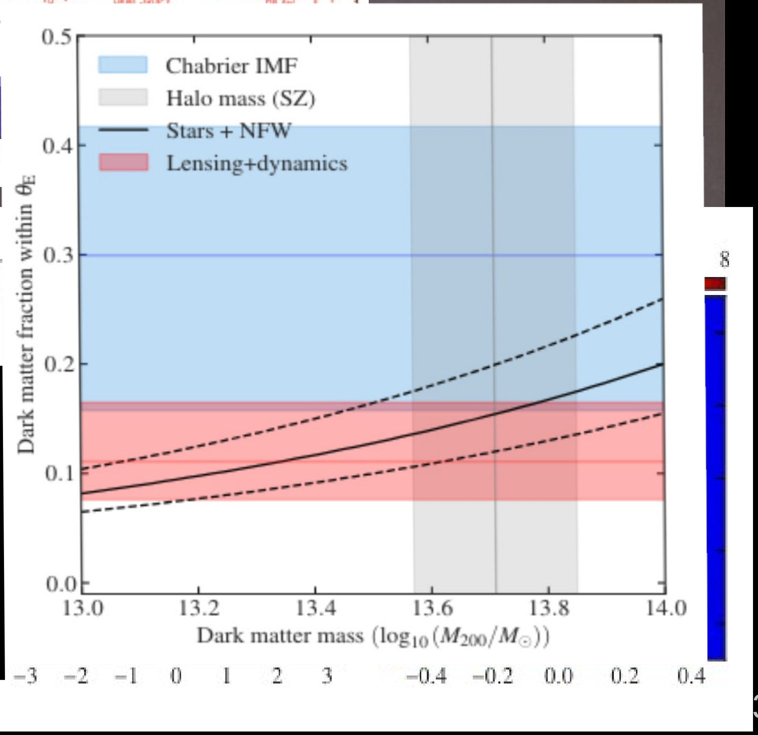


C505

_a_stunning_Einstein_ring





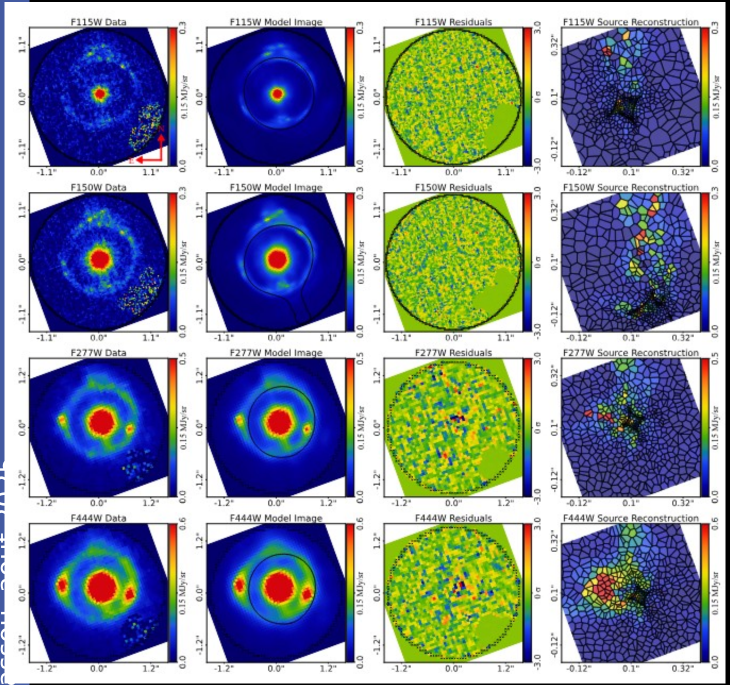


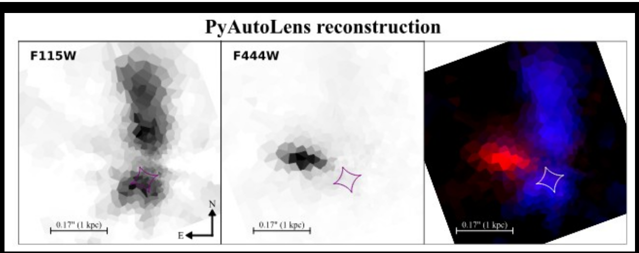
Tight constraints on the total mass

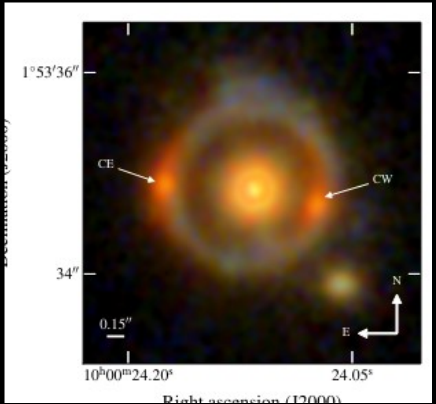
High stellar M/L Favouring bottom-heavier than Chabrier IMF in ellipticals or mild adiabatic contraction of DM halos.

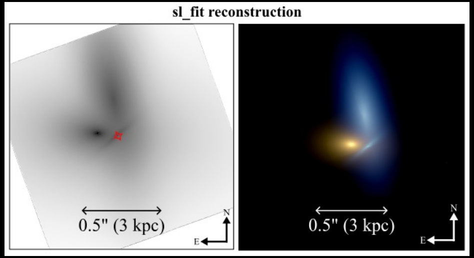
Cosmos-Web ring

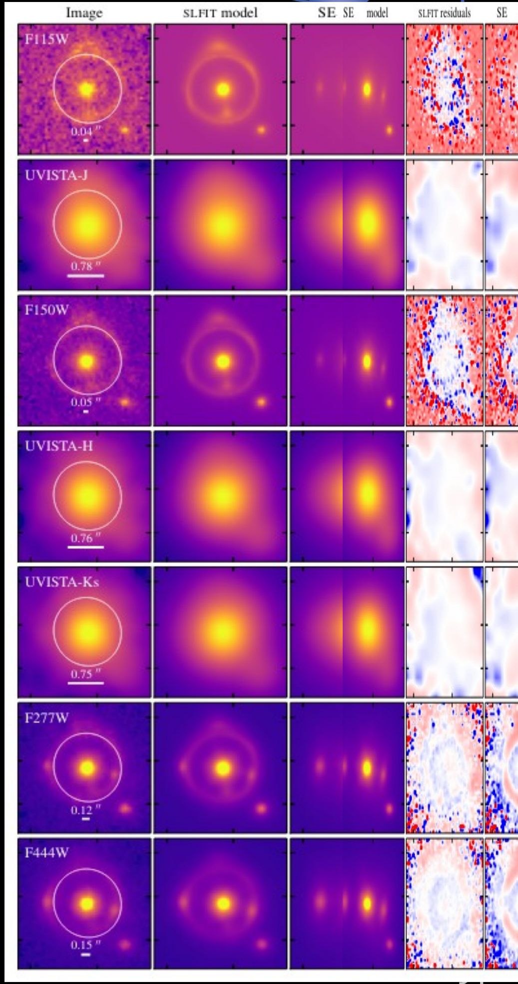
Mercier+24











Ongoing revamping of sl_fit into julia

- 100% Parametric fit
  - Simple mass model, one Sersic, one band
  - Optim.jl w/ and w/o AD
  - GPU offloading: quite easy with CUDA.jl and Flux.jl
  - Posterior sampling with Turing.jl

- Regularized source
  - Hyperparameters, source resolution
  - Basis functions: pixels, Shapelets, Sersiclets (Gram-Schmidt orthon.), wavelets...
  - Gaussian processes
  - Hybrid Sersic + flexible
  - Sparse matrices and/or massive gradients with AD?!?

Ongoing revamping of sl_fit into julia

- Hands-on!?!
  - Download tarball and go to dir
  - Julia –project=”.”
  -] instantiate

    slow… but
  -

```
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